Kinetic Magnetohydrodynamics in a Rotating Stratified Medium, With Applications to Hot Dilute Magnetized Accretion

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ABSTRACT

The drift-kinetic equation (Kulsrud 1983, 2005) is an approximation to the Boltzmann equation that is ideally suited for the study of dilute magnetohydrodynamic (MHD) astrophysical plasmas. In this paper we derive a form of the collisionless drift-kinetic equation for rotating plasmas in the limit that the sound speed is smaller than the orbital speed, and in which the magnetic fields are of subthermal strength. We construct an equilibrium profile of the pressure and magnetic fields within a rotating disk. We then apply these results to specific instabilities expected to be important in describing the flow properties of hot dilute accretion flows around black holes.

1. Introduction

Within the recent past, much progress has been made in characterizing the important dynamics of accretion flows. The magnetorotational instability (Velikhov 1959; Chandrasekhar 1960) has been applied to accretion disks (Balbus & Hawley 1991) and been shown to drive MHD fluid turbulence that can provide an outward angular momentum flux and mass accretion rate consistent with astrophysical observations, as demonstrated in a variety of numerical simulations (Hawley et al. 1996; Wardle & Ng 1999; Sano & Stone 2002; De Villiers & Hawley 2003; Fromang et al. 2004). However, there exists observational evidence of hot dilute flows, in accretion about dim mass-starved supermassive black holes, for which the mean free path is of the order of the system scale or larger. Chandra X-ray observations by Baganoff et al. (2003) have resolved the inner 1" around the Sagittarius A central black hole and demonstrated that the ion mean free path at its capture radius is only a few times smaller than the system scale. The unambiguous detection of Faraday rotation in the high-frequency radio emission about Sagittarius A (Aitken et al. 2000; Bower et al. 2003; Marrone

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et al. 2006) implies that the magnetic field is very easily strong enough to anisotropize the plasma. Estimates of mass accretion from the ambient conditions about this object overestimate its bolometric luminosity by approximately five orders of magnitude (Narayan 2002), implying that very little of the gravitational energy produced by mass accretion is radiated; the flow is radiatively inefficient, which implies and is consistent with a dilute plasma in which the bulk of the thermal energy lies with the protons.

A formulation of magnetized plasma dynamics that is especially well-suited for collisionless or mildly collisional MHD plasma equilibrium and dynamics is that of Kulsrud's drift-kinetic approximation to the Boltzmann equation (Kulsrud 1983, 2005). To lowest order the particle distribution function is characterized by dynamics only along magnetic field lines, MHD conditions of quasineutrality and zero current, and conservation of magnetic moment for particle distributions; electromagnetic fields are associated with higher-order moments of the distribution function that need not be explicitly solved. Furthermore, in these dilute plasmas, momentum and energy transfer processes such as temperature equilibration or electric resistivity that cannot be modeled through the Kulsrud formalism are not physically relevant. such effects may be modelled by a distribution function expansion in collisional frequency, and applying a more accurate collisional operator, as done in Braginskii (1965) or in Chang & Callen (1992).

The drift-kinetic equation has found use in treatments of accretion about dilute rotating astrophysical plasmas (Quataert et al. 2002; Sharma et al. 2003). However, these derivations are relatively opaque in that effects associated with the rotation of these plasmas is not explicitly taken into account. Effects peculiar to the fact that these flows are geometrically thick, where thermal speeds are subdominant but not negligible to orbital speeds and where additional effects associated with disk radial stratification must be considered (Islam & Balbus 2006), have also been ignored. In this paper we derive a drift-kinetic equation for rotating plasmas, with a simplified collision operator Bhatnager et al. (1954) that reproduces the qualitative form of viscous and thermal transport in magnetized plasmas in the limit of high collisionality (see, e.g., Snyder et al. (1997)).

For this problem we consider the following hierarchy of scales: $1/T < \omega_{pi} \ll \Omega_{ci}$, $1/L < \omega_{pi}/c \ll \rho_i$, where ω_{pi} is the ion plasma frequency, Ω_{ci} is the ion gyrofrequency, ρ_i is the ion gyroperiod, and ω_{pi}/c is the inverse ion inertial depth, and L and T are the shortest length and fastest time scales associated with this system. We also consider nonrelativistic MHD, hence with Alfvén velocities smaller than the speed of light. The gravitational acceleration is purely due to that of the central object. We consider a plasma equilibrium such that pressures parallel and perpendicular to the magnetic field are equal, hence the equilibrium particle distribution for electrons and ions has one temperature. We formulate the problem in a cylindrical geometry, where the axis of rotational lies along the vertical axis. $\hat{\boldsymbol{R}}$, $\hat{\boldsymbol{\phi}}$, and $\hat{\boldsymbol{z}}$ are unit vectors in the radial, azimuthal, and vertical directions, respectively.

The organization of this paper is as follows: in §2, we derive the drift-kinetic equation in a rotating frame in which the sound speed is subdominant (but possibly of the same order) as the orbital speed, as well as derive velocity moments that reduce to the fluid equations. In §3 justify and modify turbulent and average wave quantities appropriate to characterize accretion (see, e.g., Balbus & Hawley (1998)) to take into account the dilute and geometrically thick nature of the flow. In §4 we consider the stability of hot dilute rotating plasmas to the collisionless MRI and MTI as well as demonstrate heat fluxes and Reynolds stress, calculated in §3, are of the right form to drive accretion in this dilute thick flow. In §5 we summarize our main results as well as describe directions for further research.

2. The Drift Kinetic Equation in Rotating Frame

We begin with the Boltzmann equation, where f_s is the particle distribution function, $C\left[f_s\right]$ is a collision operator acting on f_s, $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, $\mathbf{F}_s$ is the force acting on a particle of species s, and m_s and Z_s is the mass and charge of a particle of species s:

$$\frac{\partial f_s}{\partial t} + \mathbf{v}\cdot\nabla f_s + \frac{Z_s e}{m_s}\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right)\cdot\frac{\partial f_s}{\partial \mathbf{v}} + \frac{\mathbf{F}_s}{m_s}\cdot\frac{\partial f_s}{\partial \mathbf{v}} = C\left[f_s\right], \tag{1}$$

In a dilute magnetized plasma, in which all time scales are longer than the gyroperiod and all length scales are larger than a gyroradius of a given plasma species, a natural ordering of the particle distribution function is in powers of $\Omega_s T \ll 1$, where $\Omega_s = Z_s eB/\left(m_s c\right)$ the cyclotron frequency:

$$f_s = f_s^0 + f_s^1 + \ldots \tag{2}$$

The first to employ this formalism in describing the adiabatic response of such magnetized plasmas was Chew et al. (1956); however, such a treatment is relevant only to modes with phase velocities much faster than the sound speed.

We can represent the electric field in the following manner:

$$\mathbf{E} = -\frac{1}{c}R\Omega(R)\hat{\boldsymbol{\phi}}\times\mathbf{B} - \frac{1}{c}\mathbf{u}\times\mathbf{B} + E_{\parallel}\mathbf{b} + E_R'\hat{\boldsymbol{R}} + E_z'\hat{\boldsymbol{z}}, \tag{3}$$

Here R is the radial coordinate, Ω is the angular flow velocity, $\mathbf{u}$ is the fluid bulk velocity relative to the equilibrium flow $R\Omega\hat{\boldsymbol{\phi}}$, and $E_{\parallel}$ is the electrostatic field that in an inertial frame that preserves charge neutrality, and $\mathbf{b} = \mathbf{B}/\sqrt{\mathbf{B}^2}$ is the unit vector along the magnetic field. Additional radial electric fields E'_R and E'_z arise from force balance, quasineutrality, and equal equilibrium velocity for all particle species and arise from rotation and have not been included explicitly in more simplified treatments of collisionless MHD plasmas or in simplified derivations of the drift-kinetic equation.

One must consider that to lowest order, the largest component of the electric field (provided that $\mathbf{u} \neq 0$) is the MHD electric field $-\frac{1}{c}\mathbf{u} \times \mathbf{B}$. First, $\mathbf{E} \times \mathbf{B}$ drifts arising from equilibrium E'_R and E'_z are at best of order Ω/Ω_{ci}, where $\Omega_{ci} = eB/(m_i c)$ is the ion cyclotron frequency, relative to the equilibrium flow velocity $R\Omega(R)\hat{\boldsymbol{\phi}}$ and in our analysis may be ignored. Second, the evolution of the magnetic field is described by the following MHD induction equation due to the lowest-order electric fields:

$$\begin{aligned}\frac{\partial \mathbf{B}}{\partial t} =& \nabla \times \left(R\Omega\hat{\boldsymbol{\phi}} \times \mathbf{B} + \mathbf{u} \times \mathbf{B} \right) = -\mathbf{u} \cdot \nabla \mathbf{B} - \mathbf{B} \left(\nabla \cdot \mathbf{u} \right) + \mathbf{B} \cdot \nabla \mathbf{u} + \\ & R\hat{\boldsymbol{\phi}} \mathbf{B} \cdot \nabla \Omega - \Omega \frac{\partial \mathbf{B}}{\partial \phi},\end{aligned} \tag{4}$$

In §2.1 we demonstrate the equilibrium configuration for an ion-proton plasma to derive the expression for the equilibrium angular velocity $\Omega(R)$, and in §2.2 we derive evolution equations for the zeroth-order distribution function for each species. In §2.3 and §2.4 we derive the fluid equations for this plasma from the drift-kinetic equation.

2.1. Equilibrium Configuration For a Rotating Plasma Disk

We consider an axisymmetric equilibrium in which the magnetic thermal energy is subthermal. Hence to lowest order the magnetic field does not play a role in describing the equilibrium. We demonstrate that the equilibrium magnetic field lies along surfaces of constant angular velocity, and at the midplane it is nonradial with magnetic field given by $\mathbf{B}_0 = B_0 \left(b_{\phi 0} \hat{\boldsymbol{\phi}} + b_{z0} \hat{\boldsymbol{z}} \right)$, where B_0 is the magnitude of the magnetic field. For simplicity, we consider an electron-ion plasma where the symbol s refers to each species. In the MHD limit strong radial and vertical electric fields are induced that lead to quasineutrality and single flow velocity to lowest order in the plasma.

The gravitational acceleration felt by all particles is given by the following, where R and z are radius and height, to second order in z:

$$\frac{\mathbf{F}_s}{m_s} = -\Omega_K^2 \left(1 - \frac{3z^2}{2R^2} \right) \hat{\boldsymbol{R}} - \Omega_K^2 z \hat{\boldsymbol{z}} \tag{5}$$

Where $\Omega_K^2 = GM/R^3$ is the Keplerian orbital velocity at the disk midplane ($z = 0$).

Since we consider a fat disk, there is significant variation of the orbital velocity Ω across a disk scale. To capture the lowest-order features of the angular velocity, temperature, density, pressure, and magnetic field equilibrium profiles in height and radius it is sufficient to only go up to second order in z. First, we choose a form of the angular velocity as implied by Eq. (5):

$$\Omega(R,z) = \Omega_0(R) - \frac{1}{2}\alpha_\Omega z^2, \tag{6}$$

Eq. (4) implies the following radial magnetic field up to second order in z:

$$B_{R0} = \frac{\alpha_\Omega z}{\partial\Omega_0/\partial R} B_{z0}, \tag{7}$$

Let us assume that the ion and electron temperatures have the same spatial profile. Also let $T^0 = T_i^0 + T_e^0$ and $p^0 = n^0 k_B T^0$. In equilibrium the magnetic field line must lie along isotherms, i.e. $\mathbf{B}_0 \cdot \nabla T^0$, which implies:

$$T^0(R,z) = T_0(R)\left(1 - \frac{\alpha_\Omega z^2/2}{\partial\Omega_0/\partial R}\left(\frac{\partial \ln T_0}{\partial R}\right)\right) + \mathcal{O}\left(z^4\right), \tag{8}$$

Where $T_0(R)$ is the midplane temperature. Vertical force balance:

$$-\frac{\partial p_0}{\partial z} - (m_i + m_e)\, n^0 \Omega_K^2 z = 0, \tag{9}$$

Implies the following pressure and density profile up to second order in z, employing Eq. (8):

$$p^0(R,z) = p_0(R) - \frac{1}{2}(m_i + m_e)\, n_0(R)\Omega_K^2 z^2 + \mathcal{O}\left(z^4\right), \tag{10}$$

$$n^0(R,z) = n_0(R)\left(1 - \frac{\Omega_K^2 z^2}{2\theta} + \frac{\alpha_\Omega z^2/2}{\partial\Omega_0/\partial R}\left(\frac{\partial \ln T_0}{\partial R}\right)\right) + \mathcal{O}\left(z^4\right), \tag{11}$$

Where $\theta = k_B T_0/(m_i + m_e)$ is the squared sound speed at the midplane, and $n_0(R)$ and $p_0(R)$ are the midplane number densities and pressure. Second, radial force balance is given by the following:

$$\left(\Omega_0^2 - \Omega_0\alpha_\Omega z^2\right) R - \Omega_K^2\left(1 - \frac{3z^2}{2R^2}\right) R = \frac{1}{n^0(m_i + m_e)}\frac{\partial p^0}{\partial R}, \tag{12}$$

With Eqs. (10) and (11) we can then solve for Ω_0 and α_Ω:

$$\Omega_0 = \sqrt{\Omega_K^2 + \frac{\theta}{R^2}\left(\frac{\partial \ln p_0}{\partial \ln R}\right)}, \tag{13}$$

$$\alpha_\Omega = \frac{\Omega_K^2}{2R^2\Omega_0} \times \frac{\frac{\partial \ln p_0}{\partial \ln R} + \frac{\partial \ln n_0}{\partial \ln R}}{1 + \frac{\theta}{R^2}\left(\frac{\partial \ln p_0}{\partial \ln R}\right)\left(\frac{\partial \ln T_0}{\partial \ln R}\right)\left(\frac{\partial \Omega_0^2}{\partial \ln R}\right)^{-1}}, \tag{14}$$

The disk scale height is $H = \theta^{1/2}/\Omega_K$. For thin disks, $H \ll R$ we have that the thermal speed is much smaller than the orbital speed ($\theta \ll \Omega_0^2 R^2$); since equilibrium radial gradients have radial scale heights $\sim R$, then within such a thin disk the orbital speed $\Omega \approx \Omega_K$, the equilibrium radial magnetic field within the disk is at most of order $B_{R0} \sim (H/R)\, B_{z0}$, and isotherms to a very good approximation lie along surfaces of constant R. Within thick disks, the disk height $H \lesssim R$ and $\theta \lesssim \Omega_K R$ so one cannot neglect complications that arise within some vertical coordinate within the disk – namely, substantial vertical gradients in angular velocity and temperature, and significant equilibrium radial magnetic fields.

Note that the plasma community uses the equilibrium conditions, namely $\mathbf{B}_0 \cdot \nabla\Omega$ and $\mathbf{B}_0 \cdot \nabla T^0$, as equivalently representing the Ω and T^0 as functions of the poloidal magnetic flux Ψ of an equilibrium axisymmetric magnetic field. This formalism indirectly represents, albeit to lowest nonzero orders in z, expressions for the equilibrium magnetic field and temperature in terms of Ω.

One also finds that radial and vertical force balance for ions and electrons can be represented in the following manner:

$$\frac{Z_s e}{m_s} E'_{R0} - \Omega_K^2 R \left(1 - \frac{3z^2}{2R^2}\right) = \frac{1}{m_s n^0} \frac{\partial p_0^s}{\partial R} - \Omega^2 R, \tag{15}$$

$$\frac{Z_s e}{m_s} E'_{z0} - \Omega_K^2 z = \frac{1}{m_s n^0} \frac{\partial p_0^s}{\partial z}, \tag{16}$$

2.2. Drift Kinetic Equation

To derive the drift-kinetic equation in a rotating frame from the Boltzmann equation, Eq. (1), it is best to transform to a set of variables centered about the equilibrium flow as well as using gyromotion-centered variables. The velocity can then be represented as the following:

$$\mathbf{v} = R\Omega\hat{\boldsymbol{\phi}} + \mathbf{u}_\perp + v_\parallel \mathbf{b} + \sqrt{2\mu B}\,(\hat{\mathbf{x}}_\perp \cos\psi + \hat{\mathbf{y}}_\perp \sin\psi)\,, \tag{17}$$

Where $v_\parallel$ is the velocity parallel to the magnetic field, μ is the magnetic moment, $B = \sqrt{\mathbf{B}^2}$ is the magnitude of the magnetic field, $\hat{\mathbf{x}}_\perp$ and $\hat{\mathbf{y}}_\perp$ are mutually orthogonal vectors perpendicular to the magnetic field, and ψ is the gyroangle. $\mathbf{u}_\perp$ is the the bulk flow velocity perpendicular to the magnetic field in an MHD fluid. Thus, the gyromotion-centered variables in terms of $\mathbf{v}$ and position:

$$v_{\|} = \mathbf{v}\cdot\mathbf{b} - R\Omega b_{\phi}, \tag{18}$$

$$\mu = \frac{\left(\mathbf{v} - R\Omega\hat{\boldsymbol{\phi}} + R\Omega b_{\phi}\mathbf{b} - \mathbf{b}\left(\mathbf{v}\cdot\mathbf{b}\right) - \mathbf{u}_{\perp}\right)^{2}}{2B}, \tag{19}$$

$$\tan\psi = \frac{\hat{\mathbf{y}}_{\perp}\cdot\left(\mathbf{v} - R\Omega\hat{\boldsymbol{\phi}} - \mathbf{u}_{\perp}\right)}{\hat{\mathbf{x}}_{\perp}\cdot\left(\mathbf{v} - R\Omega\hat{\boldsymbol{\phi}} - \mathbf{u}_{\perp}\right)}, \tag{20}$$

This implies the following Jacobian transforms:

$$\frac{\partial v_{\|}}{\partial\mathbf{v}} = \mathbf{b}, \tag{21}$$

$$\frac{\partial\mu}{\partial\mathbf{v}} = \sqrt{\frac{2\mu}{B}}\left(\hat{\mathbf{x}}_{\perp}\cos\psi + \hat{\mathbf{y}}_{\perp}\sin\psi\right), \tag{22}$$

$$\frac{\partial\psi}{\partial\mathbf{v}} = -\frac{\hat{\mathbf{y}}_{\perp}\cos\psi - \hat{\mathbf{x}}_{\perp}\sin\psi}{\sqrt{2\mu B}}, \tag{23}$$

The total acceleration associated with a particle of species s (where s can refer to ions or electrons) is then given by the following, where we have included the electric field as given in Eq. (3), gravitational acceleration in Eq. (5), and Eq. (17):

$$\begin{aligned} &\frac{Z_s e}{m_s}\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right) - \left(\Omega_K^2 R\left(1 - \frac{z^2}{R^2}\right)\hat{\boldsymbol{R}} + \Omega_K^2 z\hat{\boldsymbol{z}}\right) = \\ &\left(\frac{1}{m_s}\frac{\partial p_0^s}{\partial R}\hat{\boldsymbol{R}} + \frac{1}{m_s}\frac{\partial p_0^s}{\partial z}\hat{\boldsymbol{z}} - \Omega^2 R\hat{\boldsymbol{R}}\right) + \\ &\frac{Z_s e}{m_s}\left(\delta\mathbf{E}' + \frac{1}{c}B\sqrt{2\mu B}\left(\hat{\mathbf{x}}_{\perp}\cos\psi + \hat{\mathbf{y}}_{\perp}\sin\psi\right)\times\mathbf{b}\right), \end{aligned} \tag{24}$$

To lowest order in $\Omega_s T$ in the Boltzmann equation we can see that:

$$\Omega_s\left(\sqrt{2\mu B}\left(\hat{\mathbf{x}}_{\perp}\cos\psi + \hat{\mathbf{y}}_{\perp}\sin\psi\right)\times\mathbf{b}\right)\cdot\frac{\partial f_s^0}{\partial\mathbf{v}} = 0, \tag{25}$$

Reduces to the following, if we use the velocity Jacobian transforms given by Eqs. (21), (22), and (23):

$$\Omega_s\frac{\partial f_s^0}{\partial\psi} = 0 \tag{26}$$

Hence $f_s^0 \equiv f_s^0\left(v_{\|},\mu\right)$, or that the distribution function is only a function of the parallel and magnitude of the perpendicular velocity about the equilibrium flow.

The next order in the Boltzmann equation with the condition as described in Eq. (26) is given by the following, given the form of the force balance in equilibrium, Eqs. (15) and (16) and the form of the velocity, Eq. (17):

$$\begin{gathered}
\frac{\partial f_s^0}{\partial t} + \left(R\Omega\hat{\boldsymbol{\phi}} + \mathbf{u}_\perp + v_\parallel \mathbf{b} + \sqrt{2\mu B}\left(\hat{\mathbf{x}}_\perp \cos\psi + \hat{\mathbf{y}}_\perp \sin\psi\right)\right) \cdot \nabla f_s^0 + \\
\left(\frac{Z_s e}{m_s} E_\parallel \mathbf{b} + \frac{1}{m_s}\nabla p_0^s - \Omega^2 R\hat{\boldsymbol{R}} + \frac{Z_s e}{m_s}\delta\mathbf{E}'\right) \cdot \\
\left(\mathbf{b}\frac{\partial f_s^0}{\partial v_\parallel} + \sqrt{\frac{2\mu}{B}}\left(\hat{\mathbf{x}}_\perp \cos\psi + \hat{\mathbf{y}}_\perp \sin\psi\right)\frac{\partial f_s^0}{\partial \mu}\right) = C\left[f_s^0\right],
\end{gathered} \tag{27}$$

The term whose magnitude is also of the same order as the above, $\Omega_s \partial f_s^1/\partial\psi$, is a nonconstant function of ψ and so cannot contribute to the expression for f_s^0. The simplest way to keep terms that are constant in gyrophase angle is to average over ψ in Eq. (27). Before proceeding, one can derive the Jacobians of velocity with respect to time:

$$\frac{\partial v_\parallel}{\partial t} = \left(\mathbf{u}_\perp + \sqrt{2\mu B}\left(\hat{\mathbf{x}}_\perp \cos\psi + \hat{\mathbf{y}}_\perp \sin\psi\right)\right) \cdot \frac{\partial \mathbf{b}}{\partial t}, \tag{28}$$

$$\frac{\partial \mu}{\partial t} = -\sqrt{\frac{2\mu}{B}}\left(\hat{\mathbf{x}}_\perp \cos\psi + \hat{\mathbf{y}}_\perp \sin\psi\right) \cdot \left(\frac{\partial \mathbf{b}}{\partial t} v_\parallel + \frac{\partial \mathbf{u}_\perp}{\partial t}\right) - \frac{\mu}{B}\frac{\partial B}{\partial t}, \tag{29}$$

And the Jacobians of velocity with respect to position are given by the following:

$$\nabla v_\parallel = \nabla \mathbf{b} \cdot \left(\sqrt{2\mu B}\left(\hat{\mathbf{x}}_\perp \cos\psi + \hat{\mathbf{y}}_\perp \sin\psi\right) + \mathbf{u}_\perp\right) - \mathbf{b} \cdot \nabla\left(R\Omega\hat{\boldsymbol{\phi}}\right), \tag{30}$$

$$\nabla \mu = -\sqrt{\frac{2\mu}{B}}\left(\hat{\mathbf{x}}_\perp \cos\psi + \hat{\mathbf{y}}_\perp \sin\psi\right) \cdot \left(v_\parallel \nabla \mathbf{b} + \nabla \mathbf{u}_\perp + \nabla\left(R\Omega\hat{\boldsymbol{\phi}}\right)\right), \tag{31}$$

Note that we need not compute the derivatives $\partial\psi/\partial t$ and $\nabla\psi$ since f_s^0 is independent of ψ. Thus, $\partial f_s^0/\partial t \to \partial f_s^0/\partial t + \left(\partial f_s^0/\partial v_\parallel\right)\partial v_\parallel/\partial t + \left(\partial f_s^0/\partial\mu\right)\partial\mu/\partial t$ and $\nabla f_s^0 \to \nabla f_s^0 + \nabla v_\parallel\left(\partial f_s^0/\partial v_\parallel\right) + \nabla\mu\left(\partial f_s^0/\partial\mu\right)$ and the gyroaveraged Eq. (27) reduces to the following:

$$\begin{gathered}
\frac{\partial f_s^0}{\partial t} + \left(R\Omega\hat{\boldsymbol{\phi}} + \mathbf{u}_\perp + v_\parallel \mathbf{b}\right) \cdot \nabla f_s^0 + \\
\frac{\partial f_s^0}{\partial v_\parallel}\left(\mathbf{u}_\perp \cdot \frac{\partial \mathbf{b}}{\partial t} + \left(R\Omega\hat{\boldsymbol{\phi}} + \mathbf{u}_\perp + v_\parallel \mathbf{b}\right) \cdot \left(\nabla \mathbf{b} \cdot \mathbf{u}_\perp - \mathbf{b} \cdot \nabla\left(R\Omega\hat{\boldsymbol{\phi}}\right)\right)\right) + \\
\frac{\partial f_s^0}{\partial v_\parallel} 2\mu B \left\langle \left(\hat{\mathbf{x}}_\perp \cos\psi + \hat{\mathbf{y}}_\perp \sin\psi\right)\left(\hat{\mathbf{x}}_\perp \cos\psi + \hat{\mathbf{y}}_\perp \sin\psi\right) : \nabla \mathbf{b}\right\rangle_\psi - \frac{\partial f_s^0}{\partial \mu}\left(\frac{\mu}{B}\frac{\partial B}{\partial t}\right) - \\
2\mu \frac{\partial f_s^0}{\partial \mu}\left\langle \left(\hat{\mathbf{x}}_\perp \cos\psi + \hat{\mathbf{y}}_\perp \sin\psi\right)\left(\hat{\mathbf{x}}_\perp \cos\psi + \hat{\mathbf{y}}_\perp \sin\psi\right) : \left(v_\parallel \nabla \mathbf{b} + \nabla \mathbf{u}_\perp + \nabla\left(R\Omega\hat{\boldsymbol{\phi}}\right)\right)\right\rangle_\psi + \\
\left(\frac{Z_s e}{m_s} E_\parallel + \frac{1}{m_s}\mathbf{b} \cdot \nabla p_0^s - \Omega^2 R b_R\right)\frac{\partial f_s^0}{\partial v_\parallel} = \left\langle C\left[f_s^0\right]\right\rangle_\psi
\end{gathered} \tag{32}$$

Where the term $\delta\mathbf{E}' \cdot \mathbf{b}$ got absorbed into $E_\parallel$ and the gyroaveraged quantity $\langle F\rangle_\psi = \frac{1}{2\pi}\int_0^{2\pi} F\, d\psi$. The evolution of the magnetic field magnitude $B = \mathbf{B}\cdot\mathbf{b}$, from Eq. (4):

$$\frac{1}{B}\left(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\phi}\right) B = -\frac{\mathbf{u}\cdot\nabla B}{B} - \nabla\cdot\mathbf{u} + \mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b} + Rb_\phi\mathbf{b}\cdot\nabla\Omega, \tag{33}$$

The gyroaveraged tensor:

$$\langle(\hat{\mathbf{x}}_\perp\cos\psi + \hat{\mathbf{y}}_\perp\sin\psi)(\hat{\mathbf{x}}_\perp\cos\psi + \hat{\mathbf{y}}_\perp\sin\psi)\rangle_\psi = \frac{1}{2}\left(\mathbb{I} - \mathbf{bb}\right), \tag{34}$$

Finally, we use a simplified form of the collision operator (Bhatnager et al. 1954) that can qualitatively reproduce the collisional form of the viscous stress and thermal conductivity:

$$\begin{aligned}
&C\left\langle f_s^0\right\rangle = \nu_s\left(f_s^0 - \langle f_s\rangle\right)\\
&\langle f_s\rangle = \frac{n}{\left(2\pi k_B T^s/m_s\right)^{3/2}}\exp\left(-\frac{m_s\left(v_\parallel - u_\parallel\right)^2}{2T^s} - \frac{m_s\mu B}{T^s}\right)\\
&T^s = T_\parallel^s/3 + 2T_\perp^s/3,
\end{aligned} \tag{35}$$

Where $T_\parallel^s$ and $T_\perp^s$ are the parallel and perpendicular temperatures for particle species s defined as $nk_BT_\parallel^s = p_\parallel^s$ and $nk_BT_\perp^s = p_\perp^s$. ν_s is the collision frequency of species s. After some involved algebra, one can then derive the drift-kinetic equation in covariant form:

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\phi}\right)\left(f_s^0 B\right) + \nabla\cdot\left(\left[v_\parallel\mathbf{b} + \mathbf{u}_\perp\right] f_s^0 B\right) + \frac{\partial}{\partial v_\parallel}\left(f_s^0 B\left[\frac{Z_s e}{m_s}E_\parallel + \frac{1}{m_s n^0}\mathbf{b}\cdot\nabla p_0^s\right]\right) +\\
&\frac{\partial}{\partial v_\parallel}\left(f_s^0 B\left[-\mathbf{b}\cdot\left(\left[\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\phi}\right]\mathbf{u}_\perp + \left[v_\parallel\mathbf{b} + \mathbf{u}_\perp\right]\cdot\nabla\mathbf{u}_\perp\right) + \mu B\nabla\cdot\mathbf{b} +\right.\right.\\
&\left.\left.2\Omega\hat{\boldsymbol{z}}\cdot\left(\mathbf{b}\times\mathbf{u}\right) - b_\phi R\left(\mathbf{u}_\perp + v_\parallel\mathbf{b}\right)\cdot\nabla\Omega\right]\right) = -\nu_s\left(f_s^0 B - \langle f_s\rangle B\right),
\end{aligned} \tag{36}$$

Additional terms appear explicitly in the formulation of Eq. (36) that do not appear in the normal drift-kinetic equation. First, there are terms associated with noninertial rotational accelerations along the magnetic field, $2\Omega\hat{\boldsymbol{z}}\cdot(\mathbf{b}\times\mathbf{u}) - b_\phi R\left(\mathbf{u}_\perp + v_\parallel\mathbf{b}\right)\cdot\nabla\Omega$, and accelerations along the magnetic field associated with large thermal energies , $1/\left(m_s n^0\right)\mathbf{b}\cdot\nabla p_0^s$. One can also demonstrate in a straightforward manner that an equilibrium particle distribution function of the following form:

$$f_{s0}^0 = \frac{n^0(R,z)}{\left(2\pi k_B T_0^s(R,z)/m_s\right)^{3/2}}\exp\left(-\frac{m_s v_\parallel^2}{2k_B T_0^s(R,z)} - \frac{m_s\mu B}{k_B T_0^s(R,z)}\right) \tag{37}$$

With an angular velocity Ω given by Eq. (6) with Ω_0 and α_Ω given by Eqs. (13) and (14) is a solution of Eq. (36) in equilibrium, i.e. $\partial/\partial t = \Omega\partial/\partial\phi = 0$, $\mathbf{u} = \mathbf{0}$, $E_\parallel = 0$, the equilibrium magnetic field $\mathbf{b}_0\cdot\nabla\Omega = 0$, and no heat flux along magnetic field lines $\mathbf{b}_0\cdot\nabla T_0^s = 0$.

2.3. Moments of the Drift-Kinetic Equation

Now consider moments up to third order in velocity of Eq. (36), hence up to evolution equations of the heat flux, using the formalism and logic as described in Snyder et al. (1997) for a more generic plasma. The velocity volume element $d^3\mathbf{v} = B\,d\mu\,dv_\parallel\,d\psi$. For a function that is independent of ψ, $\int F\,d^3\mathbf{v} = 2\pi\int FB\,d\mu\,dv_\parallel$. The following are the nonzero moments used in deriving fluid evolution equations from the drift-kinetic equation:

$$
\begin{aligned}
n &= 2\pi \int f_s^0 B\,d\mu\,dv_\parallel \\
n_s u_\parallel &= 2\pi \int f_s^0 v_\parallel \left(B\,d\mu\,dv_\parallel\right) \\
p_\parallel^s &= 2\pi \int m_s \left(v_\parallel - u_\parallel\right)^2 f_s^0 \left(B\,d\mu\,dv_\parallel\right) \\
p_\perp^s &= 2\pi \int m_s \mu B f_s^0 \left(B\,d\mu\,dv_\parallel\right) \\
q_\parallel^s &= 2\pi \int m_s \left(v_\parallel - u_\parallel\right)^3 f_s^0 \left(B\,d\mu\,dv_\parallel\right) \\
q_\perp^s &= 2\pi \int m_s \left(v_\parallel - u_\parallel\right) \mu B f_s^0 \left(B\,d\mu\,dv_\parallel\right) \\
r_\parallel^s &= 2\pi \int m_s \left(v_\parallel - u_\parallel\right)^4 f_s^0 \left(B\,d\mu\,dv_\parallel\right) \\
r_\times^s &= 2\pi \int m_s \left(v_\parallel - u_\parallel\right)^2 \mu B f_s^0 \left(B\,d\mu\,dv_\parallel\right) \\
r_\perp^s &= 2\pi \int m_s \mu^2 B^2 f_s^0 \left(B\,d\mu\,dv_\parallel\right),
\end{aligned}
\tag{38}
$$

Where n is the number density, $u_\parallel$ is the flow velocity parallel to the magnetic field, $p_\parallel^s$ and $p_\perp^s$ are the pressures parallel and perpendicular to the magnetic field for species s, $q_\parallel^s$ and $q_\perp^s$ are the heat fluxes parallel and perpendicular to the magnetic field. $r_\parallel^s$, $r_\times^s$, and $r_\perp^s$ are fourth-order moments of the velocity. One can demonstrate the following moments of the zeroth-order collision operator as given in Eq. (35):

$$
\begin{aligned}
&2\pi \int m_s C\left[f_s^0\right] B\, d\mu\, dv_\parallel = 0 \\
&2\pi \int m_s v_\parallel C\left[f_s^0\right] B\, d\mu\, dv_\parallel = 0 \\
&2\pi \int m_s \left(v_\parallel - u_\parallel\right)^2 C\left[f_s^0\right] B\, d\mu\, dv_\parallel = -\frac{2}{3}\nu_s \left(p_\parallel^s - p_\perp^s\right) \\
&2\pi \int m_s \mu B C\left[f_s^0\right] B\, d\mu\, dv_\parallel = -\frac{1}{3}\nu_s \left(p_\perp^s - p_\parallel^s\right) \\
&2\pi \int m_s \left(v_\parallel - u_\parallel\right)^3 C\left[f_s^0\right] B\, d\mu\, dv_\parallel = -\nu_s q_\parallel^s \\
&2\pi \int m_s \mu B \left(v_\parallel - u_\parallel\right) C\left[f_s^0\right] B\, d\mu\, dv_\parallel = -\nu_s q_\perp^s,
\end{aligned}
\tag{39}
$$

Thus, taking appropriate moments of Eq. (36) with moments of the collision operator given by Eq. (39), we have that we have the following fluid equations for continuity, parallel force balance, parallel and perpendicular pressures, and heat fluxes.

$$\left(\frac{\partial}{\partial t}+\Omega\frac{\partial}{\partial\phi}\right)n+\nabla\cdot(n\mathbf{u})=0, \tag{40}$$

$$\begin{aligned}&\left(\frac{\partial}{\partial t}+\Omega\frac{\partial}{\partial\phi}\right)u_{\parallel}+\mathbf{u}\cdot\nabla u_{\parallel}+\frac{1}{nm_s}\nabla\cdot\left(p_{\parallel}^s\mathbf{b}\right)-\frac{p_{\perp}^s}{nm_s}\nabla\cdot\mathbf{b}-2\Omega\hat{\boldsymbol{z}}\cdot(\mathbf{b}\times\mathbf{u})+\\&Rb_{\phi}\mathbf{u}\cdot\nabla\Omega+\mathbf{b}\cdot\left(\left[\frac{\partial}{\partial t}+\Omega\frac{\partial}{\partial\phi}\right]\mathbf{u}_{\perp}+\mathbf{u}\cdot\nabla\mathbf{u}_{\perp}\right)-\frac{Z_se}{m_s}E_{\parallel}-\frac{1}{n^0m_s}\mathbf{b}\cdot\nabla p_0^s=0,\end{aligned} \tag{41}$$

$$\begin{aligned}&\left(\frac{\partial}{\partial t}+\Omega\frac{\partial}{\partial\phi}\right)p_{\parallel}^s+\nabla\cdot\left(p_{\parallel}^s\mathbf{u}\right)+\nabla\cdot\left(q_{\parallel}^s\mathbf{b}\right)+2p_{\parallel}^s\left(\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}+Rb_{\phi}\mathbf{b}\cdot\nabla\Omega\right)-\\&2q_{\perp}^s\nabla\cdot\mathbf{b}=-\frac{2}{3}\left(p_{\parallel}^s-p_{\perp}^s\right),\end{aligned} \tag{42}$$

$$\begin{aligned}&\left(\frac{\partial}{\partial t}+\Omega\frac{\partial}{\partial\phi}\right)p_{\perp}^s+p_{\perp}^s\left(\nabla\cdot\mathbf{u}-\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}-Rb_{\phi}\mathbf{b}\cdot\nabla\Omega\right)+\nabla\cdot\left(p_{\perp}^s\mathbf{u}\right)+\\&\nabla\cdot\left(q_{\perp}^s\mathbf{b}\right)+q_{\perp}^s\nabla\cdot\mathbf{b}=-\frac{1}{3}\left(p_{\perp}^s-p_{\parallel}^s\right),\end{aligned} \tag{43}$$

$$\begin{aligned}&\left(\frac{\partial}{\partial t}+\Omega\frac{\partial}{\partial\phi}\right)q_{\parallel}^s+\nabla\cdot\left(q_{\parallel}^s\mathbf{u}\right)+\nabla\cdot\left(\mathbf{b}r_{\parallel}^s\right)+3\left(\frac{p_{\parallel}^s\left[p_{\parallel}^s-p_{\perp}^s\right]}{m_sn}-r_{\times}^s\right)\nabla\cdot\mathbf{b}-\\&\frac{3p_{\parallel}^s}{m_sn}\mathbf{b}\cdot\nabla p_{\parallel}^s+3q_{\parallel}^s\left(\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}+Rb_{\phi}\mathbf{b}\cdot\nabla\Omega\right)=-\nu_sq_{\parallel}^s\end{aligned} \tag{44}$$

$$\begin{aligned}&\left(\frac{\partial}{\partial t}+\Omega\frac{\partial}{\partial\phi}\right)q_{\perp}^s+\nabla\cdot\left(q_{\perp}^s\mathbf{u}\right)+\nabla\cdot\left(r_{\times}^s\mathbf{b}\right)+\left(\frac{p_{\perp}^s\left[p_{\perp}^s-p_{\parallel}^s\right]}{m_sn}+r_{\times}^s-r_{\perp}^s\right)\nabla\cdot\mathbf{b}-\\&\frac{p_{\perp}^s}{m_sn}\mathbf{b}\cdot\nabla p_{\parallel}^s+q_{\perp}^s\nabla\cdot\mathbf{u}=-\nu_sq_{\perp}^s,\end{aligned} \tag{45}$$

With the following variable substitutions:

$$\begin{aligned}p^s&=\frac{1}{3}\left(p_{\parallel}^s+2p_{\perp}^s\right)\\p_v^s&=p_{\parallel}^s-p_{\perp}^s\\q^s&=q_{\parallel}^s/2+q_{\perp}^s\\q_v^s&=q_{\parallel}^s-q_{\perp}^s,\end{aligned} \tag{46}$$

One may rearrange equations for pressure evolution, Eqs. (42) and (43), into the following:

$$\frac{3}{2}\left(\frac{\partial}{\partial t}+\Omega\frac{\partial}{\partial\phi}+\mathbf{u}\cdot\nabla\right)p^s+\frac{5}{2}p^s\nabla\cdot\mathbf{u}=-\nabla\cdot\left(q^s\mathbf{b}\right)- \\ p_v^s\left(\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}-\frac{1}{3}\nabla\cdot\mathbf{u}+Rb_\phi\mathbf{b}\cdot\nabla\Omega\right) \tag{47}$$

$$\left(\frac{\partial}{\partial t}+\Omega\frac{\partial}{\partial\phi}+\mathbf{u}\cdot\nabla+\frac{4}{3}\nabla\cdot\mathbf{u}+\left[\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}+Rb_\phi\mathbf{b}\cdot\nabla\Omega\right]+\nu_s\right)p_v^s= \\ -3p^s\left(\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}+Rb_\phi\mathbf{b}\cdot\nabla\Omega-\frac{1}{3}\nabla\cdot\mathbf{u}\right)-\nabla\cdot\left(q_v^s\mathbf{b}\right)+\left(2q^s-q_v^s\right)\nabla\cdot\mathbf{b} \tag{48}$$

One may also rearrange Eqs. (42) and (43), employing Eqs. (33) and (40), into the following form first noted by Chew et al. (1956):

$$\rho B\left(\frac{\partial}{\partial t}+\Omega\frac{\partial}{\partial\phi}\right)\left(\frac{p_\perp^s}{\rho B}\right)+\rho B\mathbf{u}\cdot\nabla\left(\frac{p_\perp^s}{\rho B}\right)=-\nabla\cdot\left(q_\perp^s\mathbf{b}\right)-q_\perp^s\nabla\cdot\mathbf{b}-\frac{1}{3}\nu_s\left(p_\perp^s-p_\parallel^s\right), \tag{49}$$

$$\frac{\rho^3}{B^2}\left(\frac{\partial}{\partial t}+\Omega\frac{\partial}{\partial\phi}\right)\left(\frac{p_\parallel^s B^2}{\rho^3}\right)+\frac{\rho^3}{B^2}\mathbf{u}\cdot\nabla\left(\frac{p_\parallel^s B^2}{\rho^3}\right)=-\nabla\cdot\left(q_\parallel^s\mathbf{b}\right)-2q_\perp^s\nabla\cdot\mathbf{b}-\frac{2}{3}\nu_s\left(p_\parallel^s-p_\perp^s\right) \tag{50}$$

And the equations for the heat flux, Eqs. (44) and (45), are given by the following:

$$\left(\frac{d}{dt}+\frac{5}{3}\nabla\cdot\mathbf{u}+\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}+Rb_\phi\mathbf{b}\cdot\nabla\Omega+\nu_s\right)q^s+ \\ \left(\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}-\frac{1}{3}\nabla\cdot\mathbf{u}+Rb_\phi\mathbf{b}\cdot\nabla\Omega\right)q_v^s=-\nabla\cdot\left(\mathbf{b}\left[\frac{1}{2}r_\parallel^s+r_\times^s\right]\right)+ \\ \left(\frac{1}{2}r_\times^s+r_\perp^s\right)\nabla\cdot\mathbf{b}+\frac{5p^s}{2m_sn}\mathbf{b}\cdot\nabla p^s+\frac{5p^s}{3m_sn}\mathbf{b}\cdot\nabla p_v^s+\frac{5p_v^s}{3m_sn}\mathbf{b}\cdot\nabla\left(p^s+\frac{2}{3}p_v^s\right)- \\ \frac{\left(p^s+\frac{8}{3}p_v^s\right)p_v^s}{2m_sn}\nabla\cdot\mathbf{b}, \tag{51}$$

$$\left(\frac{\partial}{\partial t}+\Omega\frac{\partial}{\partial\phi}+2\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}+2Rb_\phi\mathbf{b}\cdot\nabla\Omega+\frac{1}{3}\nabla\cdot\mathbf{u}+\nu_s\right)q_v^s+ \\ 2\left(\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}+Rb_\phi\mathbf{b}\cdot\nabla\Omega-\frac{1}{3}\nabla\cdot\mathbf{u}\right)q^s=-\nabla\cdot\left(q_v^s\mathbf{u}\right)-\nabla\cdot\left(\mathbf{b}\left[r_\parallel^s-r_\times^s\right]\right)+ \\ \left(4r_\times^s-r_\perp^s\right)\nabla\cdot\mathbf{b}+\frac{2p_s}{m_sn}\mathbf{b}\cdot\nabla p^s+\frac{7p_v^s}{3m_sn}\mathbf{b}\cdot\nabla\left(p^s+\frac{2}{3}p_v^s\right), \tag{52}$$

In the limit of sufficiently high collisionality, one can consider a subsidiary fluid ordering of the plasma distribution function, namely of the form:

$$f_s^0=\frac{n}{\left(2\pi k_BT^s/m_s\right)^{3/2}}\exp\left(-\frac{m_sv_\parallel^2+2m_s\mu B}{2k_BT}\right)+\delta_1f_s^0+\ldots \tag{53}$$

Where $T^s_\parallel \approx T^s_\perp = T^s$ is the temperature of species s. $\delta_1 f^0_s$ refers to deviations of the zeroth-order distribution function from Maxwellian of order ν_s^{-1}. The fourth-order moments of the distribution function are then given by, to lowest order in ν_s^{-1}:

$$r^s_\parallel = 3nk_BT^s\frac{k_BT^s}{m_s}, \tag{54}$$

$$r^s_\times = nk_BT^s\frac{k_BT^s}{m_s}, \tag{55}$$

$$r^s_\perp = 2nk_BT^s\frac{k_BT^s}{m_s}, \tag{56}$$

From Eqs. (48), (51), and (52) the viscous pressure p^s_v and heat fluxes $q^s_\perp$ and $q^s_\parallel$ are given by the following to lowest order in ν_s^{-1}:

$$p^s_v \approx -\frac{3p^s}{\nu_s}\left(\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b} + Rb_\phi\mathbf{b}\cdot\nabla\Omega - \frac{1}{3}\nabla\cdot\mathbf{u}\right), \tag{57}$$

$$q^s \approx -\frac{5nk_BT^s}{2m_s\nu_s}\mathbf{b}\cdot\nabla\left(k_BT^s\right), \tag{58}$$

$$q^s_v \approx -\frac{2nk_BT^s}{m_s\nu_s}\mathbf{b}\cdot\nabla\left(k_BT^s\right), \tag{59}$$

These differ from the expressions for the viscous pressure and thermal conductivity as given in Braginskii (1965) by factors only of order unity.

2.4. Full Force Balance

Note that Eq. (36) can only describe force balance parallel to the magnetic field. In order to describe total force balance, we consider Eq. (1) with the non-MHD electric field defined in the following manner:

$$\Delta\mathbf{E} = E_\parallel\mathbf{b} + \delta\mathbf{E}' \tag{60}$$

Then to first order in the distribution function we have that:

$$\begin{aligned}&\frac{\partial f^0_s}{\partial t} + \mathbf{v}\cdot\nabla f^0_s + \left(\frac{Z_se}{m_s}\Delta\mathbf{E} - \Omega^2 R + \frac{1}{m_sn^0}\mathbf{b}\cdot\nabla p^s_0\right)\cdot\frac{\partial f^0_s}{\partial\mathbf{v}} + \\ &\frac{Z_se}{m_s}\left(-\frac{1}{c}\mathbf{u}\times\mathbf{B} - \frac{1}{c}R\Omega\hat{\boldsymbol{\phi}}\times\mathbf{B} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right)\cdot\frac{\partial f^1_s}{\partial\mathbf{v}} = C\left[f^0_s\right],\end{aligned} \tag{61}$$

Using the following transformation of velocity variables:

$$\mathbf{v} = \boldsymbol{\sigma} + R\Omega\hat{\boldsymbol{\phi}} + \mathbf{u} \tag{62}$$

Therefore, Eq. (61) reduces to the following:

$$\left(\frac{\partial}{\partial t}+\Omega\frac{\partial}{\partial \phi}\right) f_s^0+\mathbf{u}\cdot\nabla f_s^0+\boldsymbol{\sigma}\cdot\nabla f_s^0+\frac{Z_s e}{m_s c}\left(\boldsymbol{\sigma}\times\mathbf{B}\right)\cdot\frac{\partial f_s^1}{\partial\boldsymbol{\sigma}}-\frac{\partial f_s^0}{\partial\boldsymbol{\sigma}}\cdot\left(\frac{\partial\mathbf{u}}{\partial t}+\right.$$
$$\left.\left(R\Omega\hat{\boldsymbol{\phi}}+\boldsymbol{\sigma}+\mathbf{u}\right)\cdot\nabla\left(R\Omega\hat{\boldsymbol{\phi}}+\mathbf{u}\right)-\frac{Z_s e}{m_s}\Delta\mathbf{E}+\Omega^2 R\hat{\boldsymbol{R}}+\frac{1}{m_s n^0}\nabla p_0^s\right)=0 \tag{63}$$

With the following moments of the distribution function in terms of the distribution function:

$$\begin{aligned}
&\int f_s^0\, d^3\boldsymbol{\sigma}=n\\
&\int f_s^0\boldsymbol{\sigma}\, d^3\boldsymbol{\sigma}=\mathbf{0}\\
&m_s\int f_s^0\boldsymbol{\sigma}\boldsymbol{\sigma}\, d^3\boldsymbol{\sigma}=\mathbb{P}^s=p_\perp^s\mathbb{I}+\left(p_\parallel-p_\perp\right)\mathbf{bb},
\end{aligned} \tag{64}$$

Taking the moment of Eq. (63) with respect to $\boldsymbol{\sigma}$, with Eq. (64) we get:

$$n\left(\left[\frac{\partial}{\partial t}+\Omega\frac{\partial}{\partial\phi}\right]\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}-2\Omega\mathbf{u}\times\hat{\boldsymbol{z}}+R\mathbf{u}\cdot\nabla\Omega\hat{\boldsymbol{\phi}}\right)+\frac{1}{m_s}\nabla\cdot\mathbb{P}^s=$$
$$\frac{1}{c}\left(\frac{Z_s e}{m_s}\int f_s^1\boldsymbol{\sigma}\, d^3\boldsymbol{\sigma}\right)\times\mathbf{B}-\frac{1}{c}\left(\mathbf{u}\times\mathbf{B}\right)\left(\int Z_s e\int f_s^1\, d^3\boldsymbol{\sigma}\right)+Z_s en\Delta\mathbf{E}+\frac{n}{n^0}\nabla p_0^s, \tag{65}$$

Noting that currents and charges appear at first order in the distribution function:

$$e\int\left(f_i^1-f_e^1\right)\boldsymbol{\sigma}\, d^3\boldsymbol{\sigma}=\mathbf{J}, \tag{66}$$

$$e\int\left(f_i^1-f_e^1\right)\, d^3\boldsymbol{\sigma}=\rho_q, \tag{67}$$

Now adding Eq. (65) for all species together we derive the MHD force balance equation:

$$\rho\left(\left[\frac{\partial}{\partial t}+\Omega\frac{\partial}{\partial\phi}\right]\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}-2\Omega\mathbf{u}\times\hat{\boldsymbol{z}}+R\mathbf{u}\cdot\nabla\Omega\hat{\boldsymbol{\phi}}\right)=\frac{1}{c}\mathbf{J}\times\mathbf{B}+\frac{n}{n_0}\nabla p^0-$$
$$\nabla\cdot\mathbb{P}, \tag{68}$$

Where $\rho=\left(m_i+m_e\right)n$, $p^0=p_e^0+p_i^0$, $\mathbb{P}=p_\perp\mathbb{I}+\left(p_\parallel-p_\perp\right)\mathbf{bb}$, $p_\parallel=p_\parallel^i+p_\parallel^e$, and $p_\perp=p_\perp^i+p_\perp^e$. One can easily show that $\mathbf{J}=\mathbf{0}$, $\mathbf{u}=\mathbf{0}$, and $p_{\perp 0}^s=p_{\parallel 0}^s=p_0^s$ is an equilibrium solution of Eq. (68). Second, dotting Eq. (68) with $\mathbf{b}$ yields the equation for force balance parallel to the magnetic field, Eq. (41). We have neglected the contribution $\rho_q\mathbf{u}$ in the above equation since our plasma is nonrelativistic – specifically that the Alfvén speed is smaller than the speed of light.

3. Turbulent and Wave Fluxes For Dilute Rotating Plasmas

One can demonstrate by selective manipulation of the moment equations (Eq. [38]), the full MHD force balance equation (Eq. [68]), and the evolution of the magnetic field strength (Eq. [33]), that one derives the evolution equation for the total energy within a disk, using methods outlined in Balbus & Hawley (1998):

$$\begin{aligned}&\left(\frac{\partial}{\partial t}+\Omega\frac{\partial}{\partial\phi}\right)\left(\frac{1}{2}\rho u^2+\frac{3}{2}p+\frac{B^2}{8\pi}\right)+\nabla\cdot\mathcal{F}_E-\rho\mathbf{u}\cdot\frac{1}{\rho_0}\nabla p_0=\\&-\frac{\partial\Omega}{\partial\ln R}W_{R\phi}-R\frac{\partial\Omega}{\partial z}W_{z\phi}-Q_-\end{aligned}\tag{69}$$

Where $\mathcal{F}_E$ is the heat flux arising from local fluctuations, $W_{R\phi}$ is the azimuthal stress, $W_{z\phi}$ is the vertical-azimuthal stress, Q_- is a radiative loss term, and $p = p^i + p^e$ is defined in Eq. (46). In the context of disk accretion theory, the above expresses the fact that energy is generated by azimuthal stresses that couple to the free energy available from radial and vertical angular velocity gradients. This energy can then be accounted for in various ways: in a classical accretion disk, the energy flux is almost wholly radiated away; in a geometrically thick accretion disk, turbulent heat fluxes are large enough to transport at least some of this viscously generated energy (Balbus & Hawley 1998; Balbus 2003). In radiatively inefficient flows (Narayan et al. 1998), viscously generated energy must be carried away by a turbulent heat flux (Balbus 2004).

The energy flux is given by the following:

$$\begin{aligned}&\mathcal{F}_E=\mathbf{u}\left(\frac{1}{2}\rho u^2+\frac{5}{2}p\right)+\frac{1}{4\pi}\mathbf{B}\times(\mathbf{u}\times\mathbf{B})+\mathbf{b}q+\\&p_v\left(\left[\mathbf{u}\cdot\mathbf{b}\right]\mathbf{b}-\frac{1}{3}\mathbf{u}\right),\end{aligned}\tag{70}$$

With the first term in the energy flux corresponds to flux of gas kinetic energy, the second to the enthalpy, and the third term corresponds to Poynting MHD flux, and $q = q^i + q^e$ and $p_v = p_v^i + p_v^e$ as defined in Eq. (46). The fourth and fifth terms correspond to contributions due to heat fluxes along the magnetic field and the viscous stress. $W_{R\phi}$ and $W_{z\phi}$ are given by the following:

$$W_{R\phi}=\rho u_R u_\phi-\frac{B_R B_\phi}{4\pi}+p_v b_R b_\phi,\tag{71}$$

$$W_{z\phi}=\rho u_z u_\phi-\frac{B_z B_\phi}{4\pi}+p_v b_z b_\phi,\tag{72}$$

The angular momentum flux can be derived from Eq. (68) and Eq. (40):

$$\left(\frac{\partial}{\partial t}+\Omega\frac{\partial}{\partial\phi}\right)\left(\rho R\left[u_\phi+R\Omega\right]\right)+$$
$$\nabla\cdot R\left(\rho\mathbf{u}\left[u_\phi+R\Omega\right]-\frac{B_\phi\mathbf{B}}{4\pi}+p_v b_\phi\mathbf{b}+\left[p_\perp+\frac{B^2}{8\pi}\right]\hat{\boldsymbol{\phi}}\right)=0, \tag{73}$$

To understand how local fluctuations about mean quantities of the form $A = A_0 + \delta A$, whether waves or turbulence, can tap into sources of energy within this rotating system, it is easiest to consider the truncated dynamics of this system by averaging vertically and azimuthally. Define the following averaged quantity:

$$\langle A\rangle=\frac{1}{H}\int_0^{2\pi}\int_{z=-\infty}^{z=\infty}A\,dz\,d\phi \tag{74}$$

And consider fluctuations which spatially average to zero, i.e. $\langle\delta A\rangle = 0$. Contributions of fluctuations appear at second order. Recall that in equilibrium $\mathbf{u}_0 = \mathbf{0}$, $p_\parallel^0 = p_\perp^0 = p_0$, $b_{R0} = 0$, $q_0 = 0$, and $q_{v,0} = 0$. Thus, the energy and angular momentum equations can then be given by the following in the absence of collisions:

$$\frac{\partial\langle L\rangle}{\partial t}+\frac{1}{R}\frac{\partial}{\partial R}\left(R^3\Omega\langle\rho u_R\rangle+R\langle W_{R\phi}\rangle\right)=0, \tag{75}$$
$$\frac{\partial\langle\mathcal{E}\rangle}{\partial t}+\frac{1}{R}\frac{\partial}{\partial R}R\langle\mathcal{F}_{ER}\rangle-\langle\rho u_R\rangle\frac{1}{\rho_0}\frac{\partial p_0}{\partial R}=-\frac{\partial\Omega}{\partial\ln R}\langle W_{R\phi}\rangle-Q_-, \tag{76}$$

Where we have ignored the flux of gas kinetic energy, that appears at third order in fluctuating quantities, and the Poynting flux, which is subdominant to the other terms in the energy flux. We have taken $W_{z\phi}$ to be an even function of height.

$$\langle L\rangle=\langle\rho R\left(u_\phi+R\Omega\right)\rangle, \tag{77}$$
$$\langle\mathcal{E}\rangle=\left\langle\frac{1}{2}\rho u^2+\frac{1}{2}p_\parallel+p_\perp+\frac{B^2}{8\pi}\right\rangle, \tag{78}$$
$$\langle W_{R\phi}\rangle=\left\langle\rho_0\delta u_R\delta u_\phi-\frac{\delta B_R\delta B_\phi}{4\pi}+\delta p_v\delta b_R b_{\phi0}\right\rangle, \tag{79}$$
$$\langle F_{ER}\rangle=\frac{5}{2}\rho_0\langle\delta u_R\delta\theta\rangle+\langle\delta q\delta b_R\rangle-\frac{1}{3}\langle\delta p_v\delta u_R\rangle \tag{80}$$

Note that the radial mass flux term $\langle\rho u_R\rangle = \langle\delta\rho\delta u_R\rangle + \rho_0 u_{R2}$, where u_{R2} is a second order steady bulk radial flow of matter with magnitude of order $|\delta\rho/\rho_0|^2$. Balbus (2003) has suggested that one ignore mass flux terms in studies of turbulent and wave transport processes in accretion disks. This assumption of "no mass flux" has been strongly suggested by the well-known result that inertial, gravitational, and acoustic waves do not transport matter within astrophysical disks. In addition, contributions from fluctuations as well as from steady-state radial flow u_{R2} of the mass flux term cannot be determined at this level of analysis.

4. Perturbed Axisymmetric Distribution Function at the Midplane

Consider an equilibrium density distribution with given by Eq. (37) with temperature approximated by Eq. (8) and density by Eq. (11). Assume axisymmetric perturbations to equilibrium quantities of the following $\delta a \propto \exp\left(ik_R R + ik_Z z + \Gamma t\right)$, define $k_\parallel = \mathbf{k}\cdot\mathbf{b}_0$, and consider a collisionless plasma $\nu_i = \nu_e = 0$. At the midplane $B_{R0} = 0$, $\partial \ln T_0^s/\partial z = 0$, and $\partial\Omega/\partial z = 0$. Eq. (36) then reduces to the following form for ions and electrons, where we assume equal scale heights of radial and vertical ion and electron temperature gradients:

$$\begin{aligned}\delta f_i = {} & \frac{m_i v_\parallel}{k_B T_0^i}\left(\frac{-ik_\parallel \mu \delta B + e\delta E_\parallel/m_i}{\Gamma + ik_\parallel v_\parallel} - \frac{\left(2\Omega + \Omega' R\right)\Gamma b_{\phi 0}\bar{B}_R + ik_\parallel v_\parallel \Omega' R b_{\phi 0}\bar{B}_R}{ik_\parallel\left(\Gamma + ik_\parallel v_\parallel\right)}\right) f_{i0}^0 - \\ & \frac{f_{i0}^0 \bar{B}_R}{ik_\parallel}\left(\frac{\partial \ln n_0}{\partial R} - \frac{3}{2}\frac{\partial \ln T_0}{\partial R} + \left(\frac{m_i \mu B_0}{k_B T_0} + \frac{m_i v_\parallel^2}{2k_B T_0^i}\right)\frac{\partial \ln T_0^i}{\partial R}\right) + \frac{\bar{B}_R v_\parallel \partial \ln p_0/\partial R}{\Gamma + ik_\parallel v_\parallel} f_{i0}^0,\end{aligned} \tag{81}$$

$$\begin{aligned}\delta f_e = {} & \frac{m_e v_\parallel}{k_B T_0^e}\left(\frac{-ik_\parallel \mu \delta B - e\delta E_\parallel/m_e}{\Gamma + ik_\parallel v_\parallel} - \frac{\left(2\Omega + \Omega' R\right)\Gamma b_{\phi 0}\bar{B}_R + ik_\parallel v_\parallel \Omega' R b_{\phi 0}\bar{B}_R}{ik_\parallel\left(\Gamma + ik_\parallel v_\parallel\right)}\right) f_{e0}^0 - \\ & \frac{f_{e0}^0 \bar{B}_R}{ik_\parallel}\left(\frac{\partial \ln n_0}{\partial R} - \frac{3}{2}\frac{\partial \ln T_0}{\partial R} + \left(\frac{m_e \mu B_0}{k_B T_0^e} + \frac{m_e v_\parallel^2}{2k_B T_0^e}\right)\frac{\partial \ln T_0}{\partial R}\right) + \frac{\bar{B}_R v_\parallel \partial \ln p_0/\partial R}{\Gamma + ik_\parallel v_\parallel} f_{e0}^0,\end{aligned} \tag{82}$$

Where $v_i = \sqrt{k_B T_0^i/m_i}$, and $v_e = \sqrt{k_B T_0^e/m_e}$. Terms with Ω arise due to the fact that the plasma is rotating; terms with equilibrium gradients of temperature, density, or pressure may drive convective and free energy gradient instabilities. We have not considered the problem of stability away from the midplane; the most significant feature of which is that of forces along the magnetic field that act on the distribution function, seen for example in collisionless damping of electrostatic waves in a nonmagnetized gravitationally stratified medium.

$\delta E_\parallel$ is the electric field that ensures quasineutrality, i.e. $\int \delta f_i^0 B\, d\mu = \int \delta f_e^0 B\, d\mu$. One can demonstrate that in the limit of dominating ion thermal energy $T_0^i \gg T_0^e$ that the electric field $\delta E_\parallel$ and electron dynamic terms (such as $\delta p^e_{\perp,\parallel}$) becomes unimportant in describing the plasma dynamics. This is the simplification employed by Quataert et al. (2002) and Sharma et al. (2003). In §A we see that the resulting dispersion relation carefully done with equal ion and electron temperatures is not significantly different from that where the ions are orders of magnitude hotter than the electrons.

We find it useful to use the following normalizations:

$$
\begin{aligned}
&x = k_{\parallel} v_A/\Omega \\
&\mathbf{y} = \mathbf{k} v_A/\Omega \\
&\gamma = \Gamma/\Omega \\
&\alpha_P = -\left(\theta^{1/2}/\Omega\right) \frac{\partial \ln p_0}{\partial R} \\
&\alpha_T = -\left(\theta^{1/2}/\Omega\right) \frac{\partial \ln T_0}{\partial R} \\
&\beta = \theta/v_A^2,
\end{aligned}
\tag{83}
$$

Here, using the induction equation Eq. (4) and the continuity equation Eq. (40), the total force balance equation, Eq. (68), is represented by the following in terms of Eq. (83):

$$
\begin{aligned}
&\gamma^2 \bar{\mathbf{B}} - \gamma^2 \mathbf{b}_0 \left(\frac{\delta\rho}{\rho} - \frac{\alpha_P - \alpha_T}{ix\beta^{1/2}} \bar{B}_R\right) + 2\frac{d\ln\Omega}{d\ln R} \bar{B}_R \hat{\boldsymbol{R}} + 2\gamma b_{\phi 0} \left(\frac{\delta\rho}{\rho} - \frac{\alpha_P - \alpha_T}{ix\beta^{1/2}} \bar{B}_R\right) \hat{\boldsymbol{R}} + \\
&2\gamma \hat{\boldsymbol{z}} \times \bar{\mathbf{B}} = \mathbf{y} x \beta \frac{\delta p_\perp}{p_0} + x^2 \beta \frac{\delta p_\parallel - \delta p_\perp}{p_0} \mathbf{b}_0 - ix\beta^{1/2} \alpha_P \frac{\delta\rho}{\rho} \hat{\boldsymbol{R}} - x^2 \bar{\mathbf{B}} + \mathbf{y} x \frac{\delta B}{B},
\end{aligned}
\tag{84}
$$

Where $\delta B/B = \bar{B}_\phi b_{\phi 0} - (k_R/k_Z)\, \bar{B}_R b_{z0}$, $\delta p_\parallel = \delta p_\parallel^i + \delta p_\parallel^e$, and $\delta p_\perp = \delta p_\perp^i + \delta p_\perp^e$. Contributions due to $\delta\rho/\rho - (\alpha_P - \alpha_T) / \left(ix\beta^{1/2}\right) \bar{B}_R$ arise from finite plasma compressibility; in the Boussinesq limit these terms are set to zero. The eigenvalue problem consists of three equations for solving $\bar{B}_R$, $\bar{B}_\phi$, and $\delta\rho/\rho$: radial force balance, azimuthal force balance, and force balance along the equilibrium magnetic field. This is demonstrated below:

$$
\begin{aligned}
&\left(\gamma^2 + x^2 \left[1 + \frac{k_R^2}{k_Z^2}\right] + 2\frac{d\ln\Omega}{d\ln R} - 2\gamma b_{\phi 0} \frac{\alpha_P - \alpha_T}{ix\beta^{1/2}}\right) \bar{B}_R - \left(2\gamma + x^2 \frac{b_{\phi 0}}{b_{z0}} \frac{k_R}{k_Z}\right) \bar{B}_\phi + \\
&\frac{\delta\rho}{\rho} \left(2\gamma b_\phi + ix\beta^{1/2} \alpha_P\right) = \frac{k_R}{k_Z b_{z0}} x^2 \beta \frac{\delta p_\perp}{p_0},
\end{aligned}
\tag{85}
$$

$$
\left(\gamma^2 b_{\phi 0} \frac{\alpha_P - \alpha_T}{ix\beta^{1/2}} + 2\gamma\right) \bar{B}_R + \left(\gamma^2 + x^2\right) \bar{B}_\phi - \gamma^2 b_{\phi 0} \frac{\delta\rho}{\rho} = x^2 b_{\phi 0} \beta \frac{\delta p_\parallel - \delta p_\perp}{p_0},
\tag{86}
$$

$$
\bar{B}_R \left(\gamma^2 \frac{\alpha_P - \alpha_T}{ix\beta^{1/2}} - \gamma^2 \frac{k_R}{k_Z} b_{z0} + 2\gamma b_{\phi 0}\right) + \gamma^2 b_{\phi 0} \bar{B}_\phi - \gamma^2 \frac{\delta\rho}{\rho} = x^2 \beta \frac{\delta p_\parallel}{p_0},
\tag{87}
$$

Where $\delta p_\perp$ and $\delta p_\parallel$ are linear functions of $\bar{B}_R$, $\bar{B}_\phi$, and $\delta\rho/\rho$. In subsequent subsections we explore the dispersion relation associated with the rotational magnetothermal and magnetoviscous instabilities. We work in the limit of small electron thermal energies, hence $\theta \to v_i$ and $\delta E_\parallel \to 0$. For the treatment of the collisionless rotational MTI, we choose a stratified medium that is convectively stable, hence one in which $\alpha_S < 0$ or equivalently $\alpha_T < \frac{2}{5}\alpha_P$.

4.1. Dispersion Relations of the Collisionless MRI and MTI

In this section we derive the dispersion relations of the collisionless MRI as done by Quataert et al. (2002) as well as the MTI, whose dispersion relation in the fluid limit has been done by Balbus (2001); Islam & Balbus (2006). We demonstrate the salient feature of these dispersion relations, namely collisionless Landau damping of long wavelength modes along the magnetic field lines, $k_{\parallel} < \Omega/v_i$.

In this limit, one can demonstrate that the parallel and perpendicular pressures and perturbed density are given by the following from Eq. (81)

$$\begin{aligned}\frac{\delta p_{\perp}}{p_0^i} =& 2\frac{\delta B}{B} + 2\pi\left(p_0^i\right)^{-1}\int \delta f_i^0 \mu B^2\, d\mu\, dv_{\parallel} = \frac{\bar{B}_R}{ik_{\parallel}}\left(\frac{\partial \ln p_0^i}{\partial R}\right)\left(R\left(i\zeta_i\right)-1\right) - \\ &\frac{2\delta B}{B}\left(R\left(i\zeta_i\right)-1\right) + \frac{2\Omega\Gamma}{k_{\parallel}^2 v_i^2}\bar{B}_R b_{\phi 0} R\left(i\zeta_i\right),\end{aligned} \tag{88}$$

$$\frac{\delta p_{\parallel}}{p_0^i} = -\frac{2\bar{B}_R}{ik_{\parallel}}\left(\frac{\partial \ln p_0^i}{\partial R}\right)\zeta_i^2 R\left(i\zeta_i\right) + \frac{2\delta B}{B}\zeta_i^2 R\left(i\zeta_i\right) + \frac{2\Omega\Gamma}{k_{\parallel}^2 v_i^2}\bar{B}_R b_{\phi 0}\left(1 - 2\zeta_i^2 R\left(i\zeta_i\right)\right), \tag{89}$$

$$\begin{aligned}\frac{\delta \rho}{\rho} =& -\frac{\delta B}{B}\left(R\left(i\zeta_i\right)-1\right) + \frac{\bar{B}_R}{ik_{\parallel}}\left(\frac{\partial \ln p_0^i}{\partial R}\right)R\left(i\zeta_i\right) - \frac{\bar{B}_R}{ik_{\parallel}}\left(\frac{\partial \ln n_0}{\partial R}\right) + \\ &\frac{2\Omega\Gamma}{k_{\parallel}^2 v_i^2}\bar{B}_R b_{\phi 0} R\left(i\zeta_i\right),\end{aligned} \tag{90}$$

Where we have transparently demonstrated the presence of terms that arise from rotation (terms proportional to Ω) and those arising from finite equilibrium gradients in density and temperature. $\zeta_i = \Gamma/\left(k_{\parallel} v_i \sqrt{2}\right)$ and $R\left(\xi\right)$ is the plasma response function, defined as the following:

$$R\left(\xi\right) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{x e^{-x^2}}{x-\xi}\, dx, \tag{91}$$

By substituting for the rotational term $2\Omega\Gamma/\left(k_{\parallel}^2 v_i^2\right)$ one can rearrange the perturbed parallel and perpendicular pressures into the following linear combination of density, $\delta B/B$, and $\bar{B}_R$:

$$\frac{\delta p_{\perp}^i}{p_0^i} = \frac{\delta \rho}{\rho} - \frac{\delta B}{B}\left(R\left(i\zeta_i\right)-1\right) + \frac{\bar{B}_R}{ik_{\parallel}}\left(\frac{\partial \ln n_0}{\partial R} - \frac{\partial \ln p_0^i}{\partial R}\right), \tag{92}$$

$$\begin{aligned}\frac{\delta p_{\parallel}^i}{p_0^i} =& \left(\frac{1 - 2\zeta_i^2 R\left(i\zeta_i\right)}{R\left(i\zeta_i\right)}\right)\frac{\delta \rho}{\rho} - \left(\frac{1 - \left[1 + 2\zeta_i^2\right]R\left(i\zeta_i\right)}{R\left(i\zeta_i\right)}\right)\frac{\delta B}{B} + \\ &\frac{\bar{B}_R}{ik_{\parallel}}\left(\frac{1 - 2\zeta_i^2 R\left(i\zeta_i\right)}{R\left(i\zeta_i\right)} \times \frac{\partial \ln n_0}{\partial R} - \frac{\partial \ln p_0^i}{\partial R}\right),\end{aligned} \tag{93}$$

In the limit of $|\zeta_i| \gg 1$, one has that:

$$R\left(i\zeta_i\right) = \frac{1}{2\zeta_i^2} - \frac{3}{4\zeta_i^4} + \frac{15}{8\zeta_i^6} + \mathcal{O}\left(1/\zeta_i^8\right), \tag{94}$$

And in the limit of $|\zeta_i| \ll 1$, one has that:

$$R\left(i\zeta_i\right) = 1 - \zeta_i\sqrt{\pi} + \mathcal{O}\left(\zeta_i^2\right), \tag{95}$$

One can then demonstrate that the pressure responses reduce to that of the double-adiabatic limit (Chew et al. 1956) in the presence of equilibrium pressure gradients, i.e.:

$$\frac{\delta p_{\|}}{p_0^i} \rightarrow \frac{\delta B}{B} + \frac{\delta\rho}{\rho} - \xi_R\frac{\partial \ln T_0}{\partial R}, \tag{96}$$

$$\frac{\delta p_{\perp}}{p_0^i} \rightarrow 3\frac{\delta\rho}{\rho} - 2\frac{\delta B}{B} + \xi_R\left(3\frac{\partial \ln n_0}{\partial R} - \frac{\partial \ln p_0^i}{\partial R}\right), \tag{97}$$

Where from Eq. (4), one can demonstrate $\bar{B}_R = ik_{\|}\xi_R$ with ξ_R being the radial fluid displacement. However, since the phase velocity of the modes are at best of order the sound speed, i.e. $|\zeta_i| \sim 1$, these perturbations are not adiabatic and the opposite, slow wave ($|\zeta|_i \ll 1$) limit, holds for most unstable wavenumbers. Here we have express the perturbed pressures only up to first order in ζ_i:

$$\frac{\delta p_{\|}}{p_0^i} \rightarrow \frac{\delta\rho}{\rho} + \sqrt{\pi}\zeta_i\frac{\delta B}{B} - \xi_R\frac{\partial \ln T_0}{\partial R}, \tag{98}$$

$$\frac{\delta p_{\perp}}{p_0^i} \rightarrow \frac{\delta\rho}{\rho} - \sqrt{\pi}\zeta_i\frac{\delta B}{B} + \xi_R\left(3\frac{\partial \ln n_0}{\partial R} - \frac{\partial \ln p_0^i}{\partial R}\right), \tag{99}$$

Dispersion relations for the collisionless MRI and MTI are displayed below:

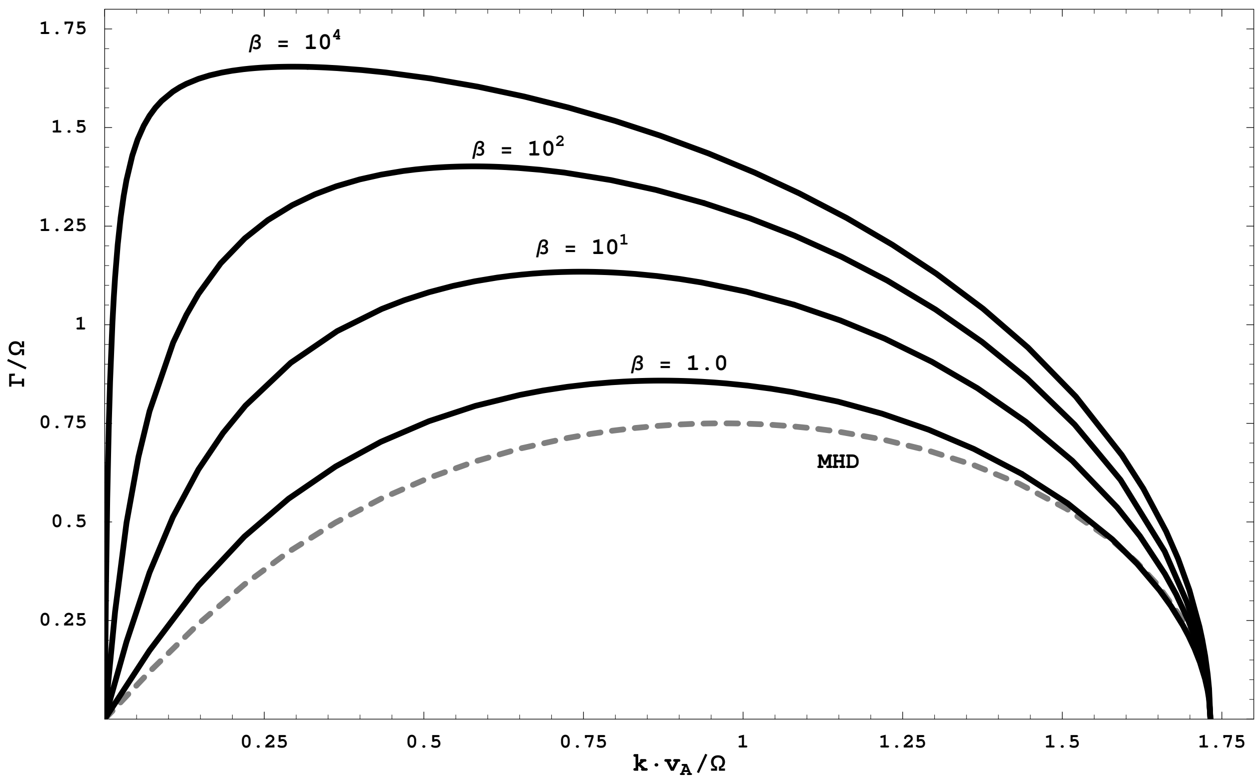

Fig. 1.— Plot of the growth rate for purely vertical wavenumbers $k_R = 0$ and no equilibrium gradients of pressure or temperature, with a Keplerian-like rotation profile, and varying β. Here, we see there's a turnover where the phase velocity of the wave becomes supersonic, roughly at wavenumbers $k_\parallel v_i \simeq \Omega$.

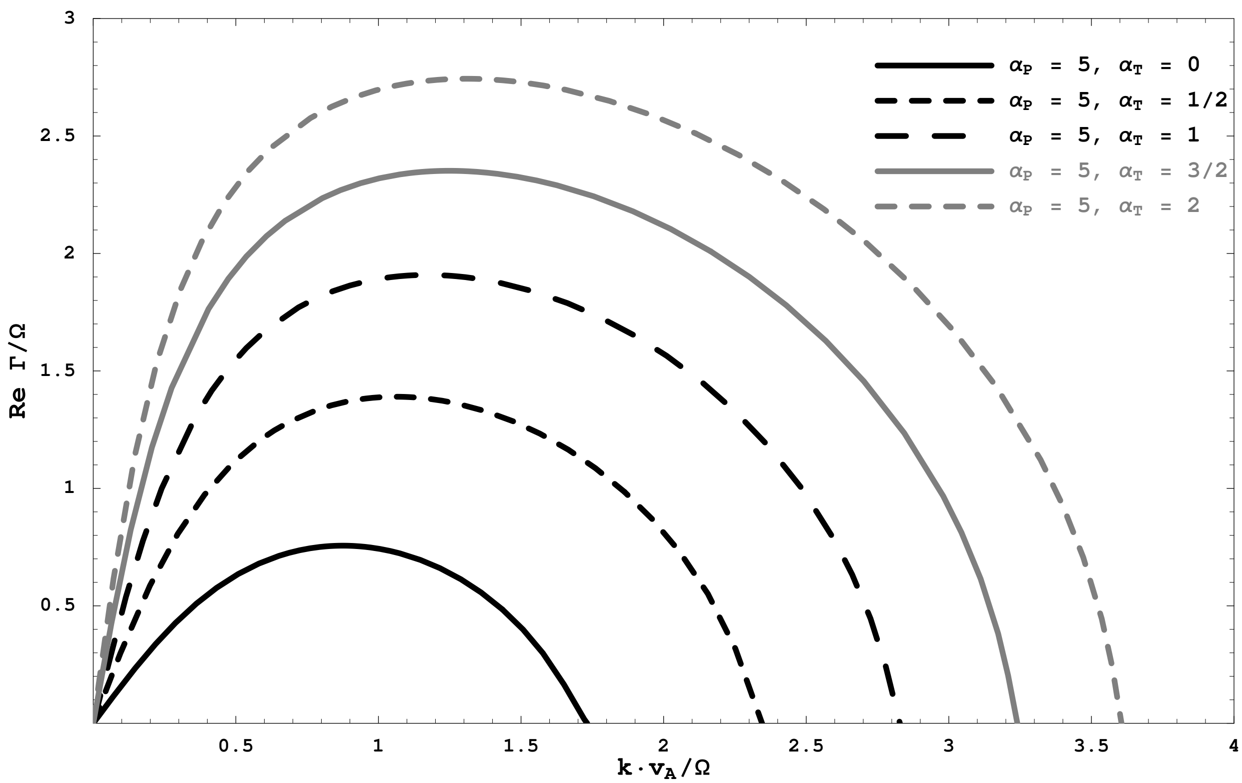

Fig. 2.— Plot of the real part of the growth rate for purely vertical wavenumbers $k_R = 0$, with a Keplerian-like rotation profile and $\beta = 10^2$ and different equilibrium gradients of pressure and temperature. Here $\alpha_P = 5$ and different $\alpha_T = 0$, such that $0 < \alpha_T < \frac{2}{5}\alpha_P$, so that the plasma remains convectively stable.

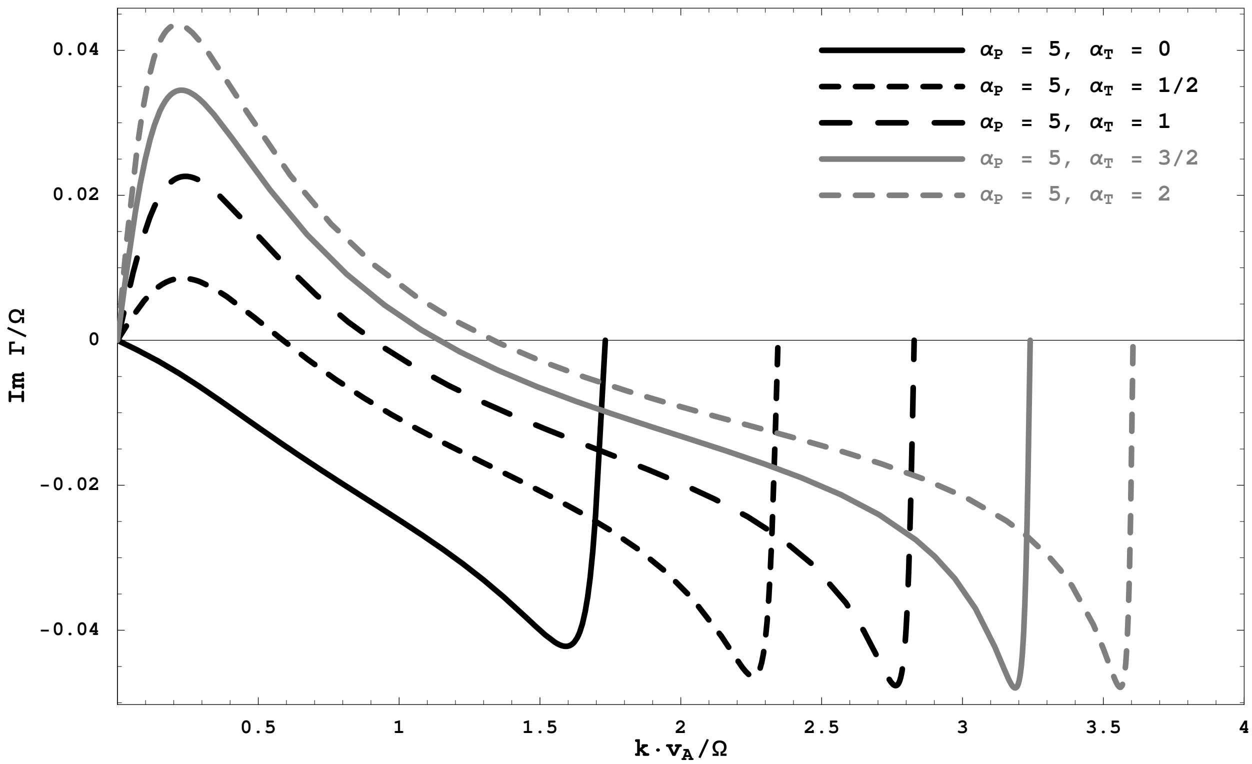

Fig. 3.— Plot of the imaginary part of the growth rate for purely vertical wavenumbers $k_R = 0$, with a Keplerian-like rotation profile and $\beta = 10^2$ and different equilibrium gradients of pressure and temperature. Here $\alpha_P = 5$ and we choose $\alpha_T < \frac{2}{5}\alpha_P$ so that the plasma is convectively stable but unstable to the magnetothermal instability.

The overarching feature of the plasma response via the MRI and MTI is that of relatively strong collisionless damping of modes along the magnetic fields for long wavelength modes $k_\| < \Omega/v_i$, such that at these wavenumbers the phase velocity remains of the order of the sound speed. This feature has been noted in previous studies of the collisionless MRI (Quataert et al. 2002; Sharma et al. 2003). This damping has the effect of suppressing pressure variations for sufficiently small wavelengths, such that as the plasma β decreases to order 1 and smaller the effects of anisotropic pressure become insignificant over much of the range of unstable wavenumbers. Shown below in Fig. (4) for the marginally convectively stable case, $\alpha_P = 5$ and $\alpha_T = 2$. Dispersion relations for the collisionless MRI and MTI

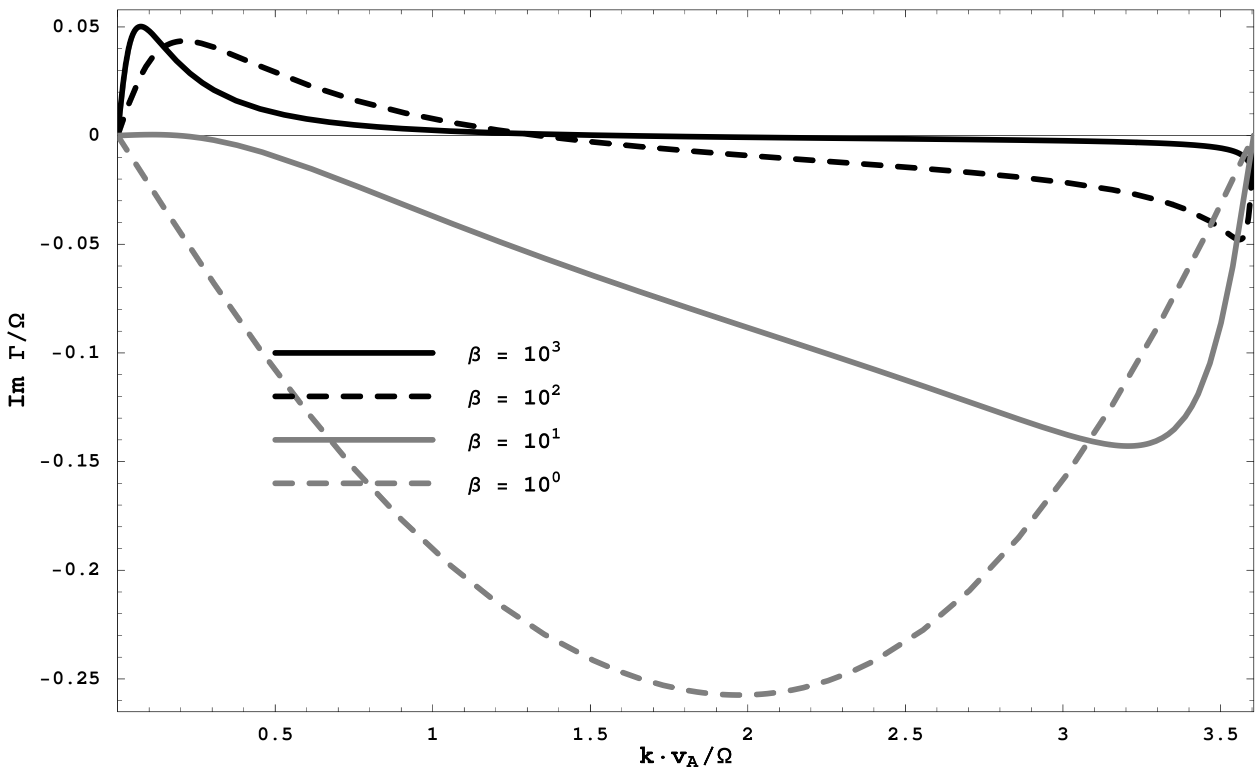

Fig. 4.— Plot of the imaginary component of the growth rate (associated with finite compressibility) for various β and a marginally convectively stable Keplerian-like rotating flow. For large β the imaginary component reaches a maximum at those wavenumbers at which the growth rate of the instability saturates. As $\beta \to 1$, $\mathrm{Im}\,\Gamma/\Omega < 0$ and its magnitude increases until it is within the same magnitude of the growth rate.

are similar to their fluid counterparts – the magnetoviscous instability, or MVI (Islam & Balbus 2005), and the magnetoviscous-thermal instability, or MVTI (Islam & Balbus 2006), respectively. Instead of collisionless damping in the case of the instabilities analyzed within this paper, in fluid treatments it is finite (but dynamically important) viscosity and thermal conductivity that plays this role.

4.2. Quadratic Fluxes of Collisionless MRI and MTI

Here, we determine the normalized quadratic heat flux, Eq. (80), and the radial azimuthal stress, Eq. (79), associated with a given mode of purely vertical wavenumber k_Z. We normalize these fluxes as a function of fixed Lagrangian radial displacement $\xi_R = \delta u_R/\Gamma$. We require expressions for the viscous pressure p_v and the heat flux q. From Eq. (81) and the moment equations, Eq. (38), we have that:

$$\delta u_\parallel/v_i = -i\zeta_i\sqrt{2}R\left(i\zeta_i\right)\left(\frac{\delta B}{B} - \frac{2\Omega\Gamma}{k_\parallel^2 v_i^2}\bar{B}_R b_{\phi 0} + \frac{i\bar{B}_R}{k_\parallel}\left(\frac{\partial \ln p_0^i}{\partial R}\right)\right) + \frac{i\bar{B}_R}{k_\parallel}b_{\phi 0}\Omega' R, \quad (100)$$

$$\begin{aligned}\delta q_\parallel/\left(p_0^i v_i\right) &= -3\delta u_\parallel/v_i + \frac{2\pi m_i}{p_0^i v_i}\int v_\parallel^3 \delta f_i^0 B\, d\mu\, dv_\parallel = \\ &\left(i\zeta_i\sqrt{2}\left(\frac{\delta B}{B}\right) - \frac{\bar{B}_R}{k_\parallel}\left(\frac{\partial \ln p_0^i}{\partial R}\right)\zeta_i\sqrt{2} - \frac{\Omega\zeta_i^2}{k_\parallel v_i}i\bar{B}_R\cos\chi\right)\left(\left[2\zeta_i^2+3\right]R\left(i\zeta_i\right)-1\right),\end{aligned} \quad (101)$$

$$\delta q_\perp/\left(p_0^i v_i\right) = -\delta u_\parallel/v_i + \frac{2\pi m_i}{p_0^i v_i}\int v_\parallel \mu B^2 \delta f_i^0\, d\mu\, dv_\parallel = -i\zeta_i\sqrt{2}\left(\frac{\delta B}{B}\right)R\left(i\zeta_i\right), \quad (102)$$

Expressions for the heat flux and radial-azimuthal stress for these axisymmetric modes at the disk midplane are given by the following:

$$W_{R\phi} = \mathrm{Re}\left(\rho_0 \delta u_R^* \delta u_\phi - v_A^2 \bar{B}_R^* \bar{B}_\phi + b_{\phi 0}\bar{B}_R^* \delta p_v\right), \quad (103)$$

$$F_{ER} = \mathrm{Re}\left(\frac{5}{2}\delta u_R^* \delta\theta - \delta q \bar{B}_R^* - \frac{1}{3}\delta p_v \bar{B}_R^*\right), \quad (104)$$

One can employ Eq. (46), with expressions for the perturbed pressures as given in Eqs. (88) and (89) and heat fluxes given in Eqs. (101) and (102). The form of the relative perturbed density and toroidal magnetic field $\delta\rho/\rho$ and $\bar{B}_\phi$ are described in the eigenvalue equations (Eqs. [85], [86], and [87]). Expressions for the relevant perturbed quantities in terms of ξ_R are described in §B. The azimuthal stress is normalized in units of $\rho\Omega^2\left|\xi_R\right|^2$ and the heat flux in terms of $\rho v_i \Omega^2 \left|\xi_R\right|^2$.

In Figs. (5) and (6) are plots of the heat flux and azimuthal stress for the collisionless MTI for different $0 < \alpha_T < \frac{2}{5}\alpha_P$.

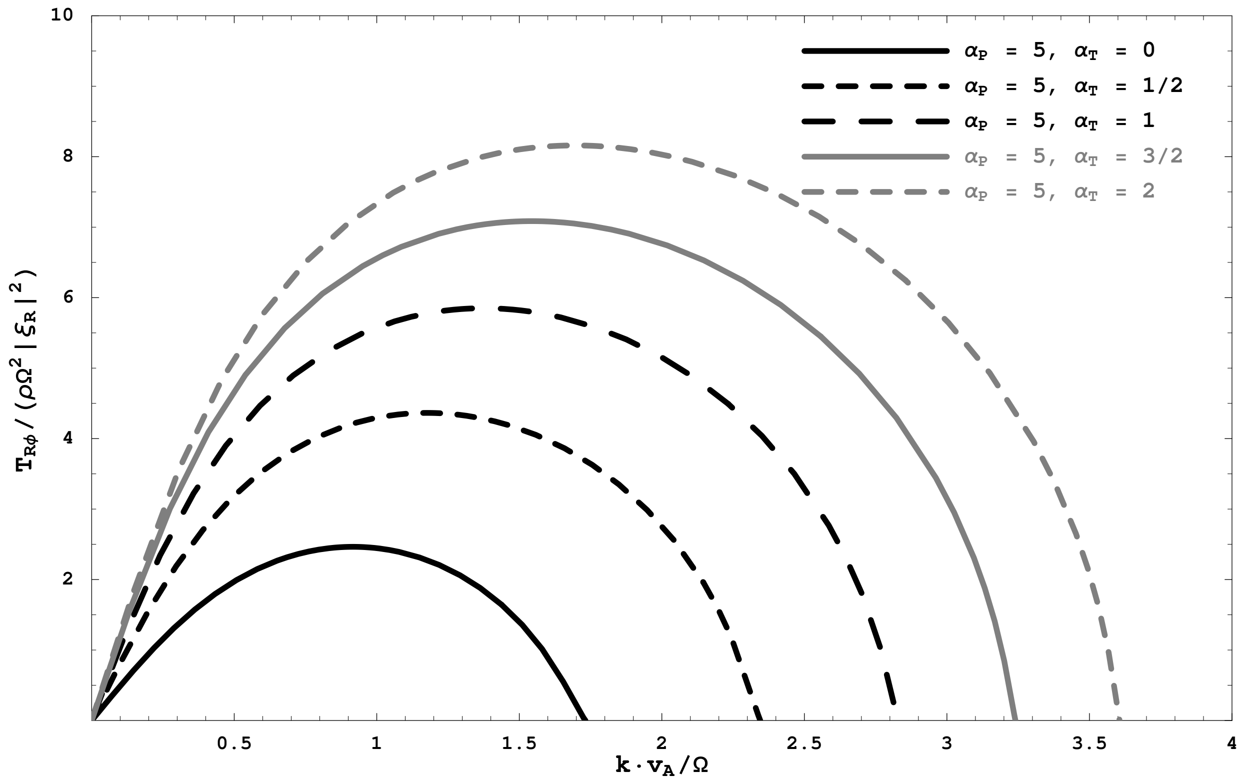

Fig. 5.— Outwards normalized azimuthal stress for the collisionless MTI, for a Keplerian-like rotation profile, $\beta = 10^2$, and $b_{\phi 0} = b_{z0} = 1/\sqrt{2}$, for various convectively stable equilibrium profiles with $\alpha_P = 5$ and $0 \leq \alpha_T \leq 2$.

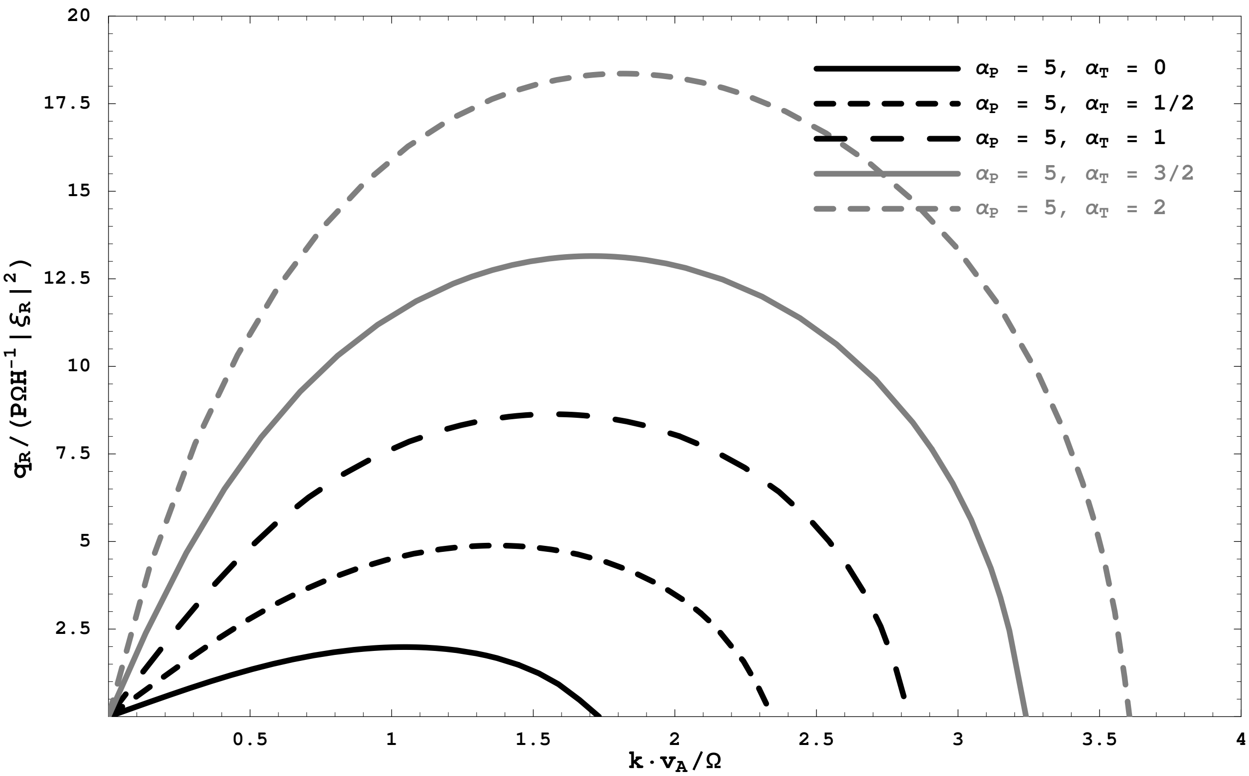

Fig. 6.— Same as Fig. (5), except for quadratic heat flux.

In Figs. (7) is a plot of the azimuthal stress for the collisionless MRI for various $\beta \geq 1$. The the heat flux for the collisionless MRI is zero. There are no equilibrium radial gradients of temperature or density, the growth rate is purely real, so that for a given mode the temperature and viscous pressure perturbations are out of phase with the perturbed radial velocity, and theperturbed heat flux is out of phase with the perturbed radial magnetic field. The salient features of these instabilities is that they produce the right type of azimuthal

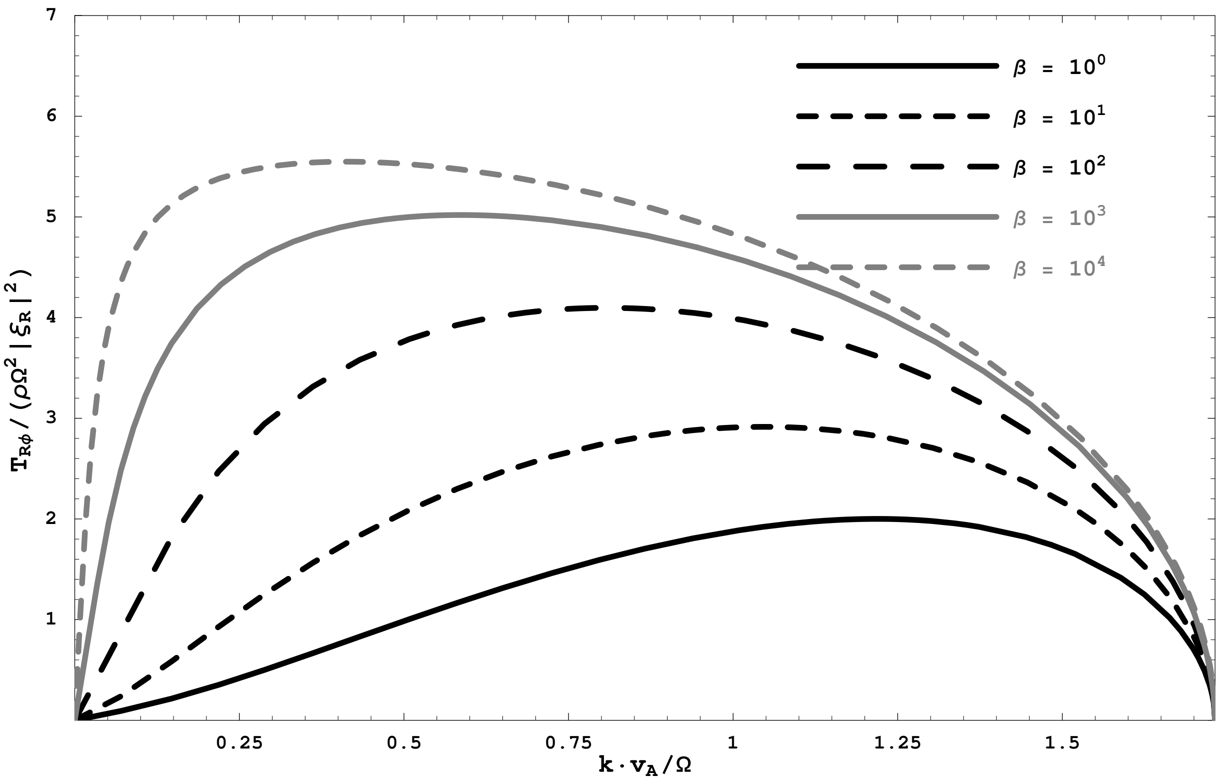

Fig. 7.— Outwards normalized azimuthal stress for the collisionless MRI, for a Keplerian-like rotation profile, $1 \leq \beta \leq 10^4$, and $b_{\phi 0} = b_{z0} = 1/\sqrt{2}$.

stress that can drive accretion. The general sense of the Reynolds stress is outwards for all unstable wavenumbers for the collisionless MTI; however, Islam & Balbus (2006) demonstrates that the MVTI can have a generally small range of small wavenumbers for even an unstable Keplerian rotational profile in which the azimuthal stress is negative.

5. Summary of Results

In this paper we have derived the drift kinetic equation explicitly in a rotating frame with possible significant gas pressures and only mild collisionality, with application to hot, dilute, weakly-magnetized (in the sense that magnetic forces are subdominant in equilibrium), at best mildly relativistic systems such as as dim accretion about supermassive black holes. We see physical terms explicitly associated with disk stratification as well as rotation. We also see that one may rather easily derive modifications of the azimuthal stress and heat flux due to fluctuations or waves in accreting systems (Balbus & Hawley 1998; Balbus 2003) due to dilute plasmas, as demonstrated in §3, in order to characterize how or whether instabilities may create the right type of turbulence that drives accretion.

We analyze the collisionless MRI and MTI, which have been demonstrated (Balbus 2001, 2004; Islam & Balbus 2005, 2006) from a fluid treatment to destabilize a plasma, through anisotropic viscosities and thermal conductivities, that possesses adverse angular velocity or temperature gradients. We demonstrate that the dispersion relation for the collisionless MRI matches that of Quataert et al. (2002), that both the collisionless MRI and MTI match with their fluid counterparts – the MVI and the MVTI, respectively. Heat fluxes and azimuthal stresses associated with them have the right sense (i.e., positive), to drive accretion in fat dilute nonradiative rotating plasmas, and roughly match their respective fluid counterparts. Unsurprisingly, we also find that we may ignore complications arising from finite electron temperature.

Although we have applied the drift-kinetic equation to a single but important class of instability in Keplerian-like rotating systems, its representation as given in Eq. (36) lends itself to much richer studies of these types of dilute plasmas. First, even if the equilibrium can be described by fluid dynamics, it may be unstable to shorter-wavelength collisionless MHD modes; as noted by Sharma et al. (2003) and as can be demonstrated by including collisional effects in the drift-kinetic equation, there exists a range of collision frequencies $\nu < \Omega\beta^{1/2}$ where the MHD dynamics are consistent with that of a collisionless plasma. Second, we have showed that for these fat magnetized plasmas there exists significant variation of equilibrium magnetic field, temperature, and angular velocity through a disk height. Significant equilibrium forces along the magnetic field, at positions away from the midplane, can then modify the behavior of disk MHD instabilities even in mildly collisional plasmas. Third, we can construct more accurate collisionless approximations to the third-order moments of the distribution function, heat fluxes of the pressures parallel and perpendicular to the magnetic field, that provide better approximations to MHD instabilities peculiar to a rotating stratified disk than those of Snyder et al. (1997).

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A. Finite Electron Temperatures

One can demonstrate that the ion and electron perturbed density are given by the following:

$$\begin{aligned}&\frac{\delta n_i}{n_0} = 2\pi n_0^{-1}\int \delta f_i B \, d\mu \, dv_\parallel + \frac{\delta B}{B} = -\frac{ie\delta E_\parallel}{k_\parallel k_B T_0^i} R\left(i\zeta_i\right) - \frac{\delta B}{B}\left(R\left(i\zeta_i\right) - 1\right) + \\ &\frac{2\Omega\Gamma}{k_\parallel^2 v_i^2} R\left(i\zeta_i\right) b_{\phi 0}\bar{B}_R + \frac{i\bar{B}_R}{k_\parallel}\left(\frac{\partial \ln n_0}{\partial R}\right) - \frac{i\bar{B}_R}{k_\parallel}\left(\frac{\partial \ln p_i^0}{\partial R}\right) R\left(i\zeta_i\right),\end{aligned} \tag{A1}$$

$$\begin{aligned}&\frac{\delta n_e}{n_0} = 2\pi n_0^{-1}\int \delta f_e B \, d\mu \, dv_\parallel + \frac{\delta B}{B} = \frac{ie\delta E_\parallel}{k_\parallel k_B T_0^e} R\left(i\zeta_e\right) - \frac{\delta B}{B}\left(R\left(i\zeta_e\right) - 1\right) + \\ &\frac{2\Omega\Gamma}{k_\parallel^2 v_e^2} R\left(i\zeta_e\right) b_{\phi 0}\bar{B}_R + \frac{i\bar{B}_R}{k_\parallel}\left(\frac{\partial \ln n_0}{\partial R}\right) - \frac{i\bar{B}_R}{k_\parallel}\left(\frac{\partial \ln p_e^0}{\partial R}\right) R\left(i\zeta_e\right),\end{aligned} \tag{A2}$$

If we make the following variable substitutions:

$$T_0^i = \left(T_0^i + T_0^e\right)\cos^2\psi, \tag{A3}$$

$$T_0^e = \left(T_0^i + T_0^e\right)\sin^2\psi, \tag{A4}$$

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Thus, with Eq. (83) and Eqs. (A3) and (A4), expressions for ζ_i and ζ_e are given by the following:

$$\zeta_i = \frac{\gamma}{x\sqrt{2\beta}} \sec\psi \left(\frac{m_i}{m_i + m_e}\right)^{1/2}, \tag{A5}$$

$$\zeta_e = \frac{\gamma}{x\sqrt{2\beta}} \csc\psi \left(\frac{m_e}{m_i + m_e}\right)^{1/2}, \tag{A6}$$

With quasineutrality $\delta n_i = \delta n_e$ one can demonstrate that the electric field is given by the following:

$$\begin{aligned}\frac{ie\delta E_\parallel}{k_\parallel (m_i + m_e)\theta} = &- \frac{\delta B}{B}\sin^2\psi\cos^2\psi \frac{R(i\zeta_i) - R(i\zeta_e)}{R(i\zeta_i)\sin^2\psi + R(i\zeta_e)\cos^2\psi} - \\ &\frac{\bar{B}_R}{ik_\parallel}\sin^2\psi\cos^2\psi \frac{\alpha_P\left(R(i\zeta_i) - R(i\zeta_e)\right)}{R(i\zeta_i)\sin^2\psi + R(i\zeta_e)\cos^2\psi} + \\ &\frac{2\Omega\Gamma}{k_\parallel^2\theta}\bar{B}_R \frac{R(i\zeta_i)\frac{m_i}{m_i+m_e}\sin^2\psi - R(i\zeta_e)\frac{m_e}{m_i+m_e}\cos^2\psi}{R(i\zeta_i)\sin^2\psi + R(i\zeta_e)\cos^2\psi},\end{aligned} \tag{A7}$$

One can then demonstrate that the total perturbed parallel and perpendicular pressures are given by the following using normalizations given by Eq. (83):

$$\begin{aligned}\frac{\delta p_\perp}{p_0} = &\frac{i\bar{B}_R}{x\beta^{1/2}}\alpha_P\left(R(i\zeta_i)\cos^2\psi + R(i\zeta_e)\sin^2\psi - 1\right) - \\ &\frac{2\delta B}{B}\left(R(i\zeta_i)\cos^2\psi + R(i\zeta_e)\sin^2\psi - 1\right) + \\ &\frac{2\gamma}{x^2\beta}\bar{B}_R b_{\phi 0}\frac{m_i R(i\zeta_i) + m_e R(i\zeta_e)}{m_i + m_e} + \frac{2ie\delta E_\parallel}{k_\parallel (m_i + m_e)\theta}\left(R(i\zeta_i) - R(i\zeta_e)\right),\end{aligned} \tag{A8}$$

$$\begin{aligned}\frac{\delta p_\parallel}{p_0} = &- \frac{2i\bar{B}_R}{x\beta^{1/2}}\alpha_P\left(\zeta_i^2 R(i\zeta_i)\cos^2\psi + \zeta_e^2 R(i\zeta_e)\sin^2\psi\right) + \\ &\frac{2\delta B}{B}\left(\zeta_i^2 R(i\zeta_i)\cos^2\psi + \zeta_e^2 R(i\zeta_e)\sin^2\psi\right) + \\ &\frac{2\gamma}{x^2\beta}\bar{B}_R b_{\phi 0}\left(1 - \frac{2\zeta_i^2 m_i R(i\zeta_i) + 2\zeta_e^2 m_e R(i\zeta_e)}{m_i + m_e}\right) + \\ &\frac{2ie\delta E_\parallel}{k_\parallel (m_i + m_e)\theta}\left(\zeta_i^2 R(i\zeta_i) - \zeta_e^2 R(i\zeta_e)\right),\end{aligned} \tag{A9}$$

Shown below are plots of the collisionless MRI (Fig. [8]) and MVTI (Fig. [9]) taken for various ratios of T_0^e/T_0^i. These are all taken for a $\beta = 10^2$ plasma, with Keplerian-like rotation profile $\Omega \propto R^{-3/2}$, and equal azimuthal and vertical equilibrium magnetic fields $b_{\phi 0} = b_{z0} = \sin\pi/4$. For the MVTI, we use a system that is convectively stable, hence $\alpha_P = 5$ and $\alpha_T = 1$.

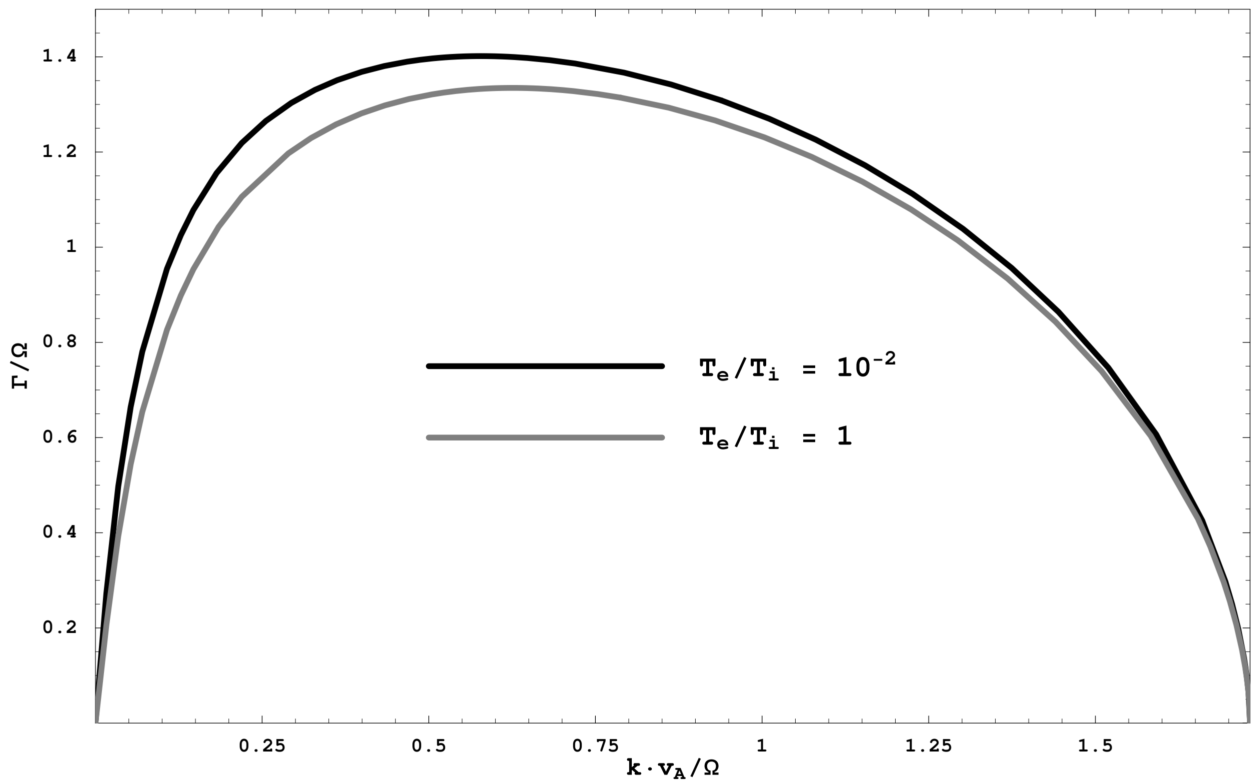

Fig. 8.— Plot of the real part of the growth rate as a function of wavenumber for the collisionless MRI for both equal and negligible ion and electron temperatures. The agreement when taking the case of equal ion and electron temperatures and one in which the electron temperature is relatively negligible are quite similar.

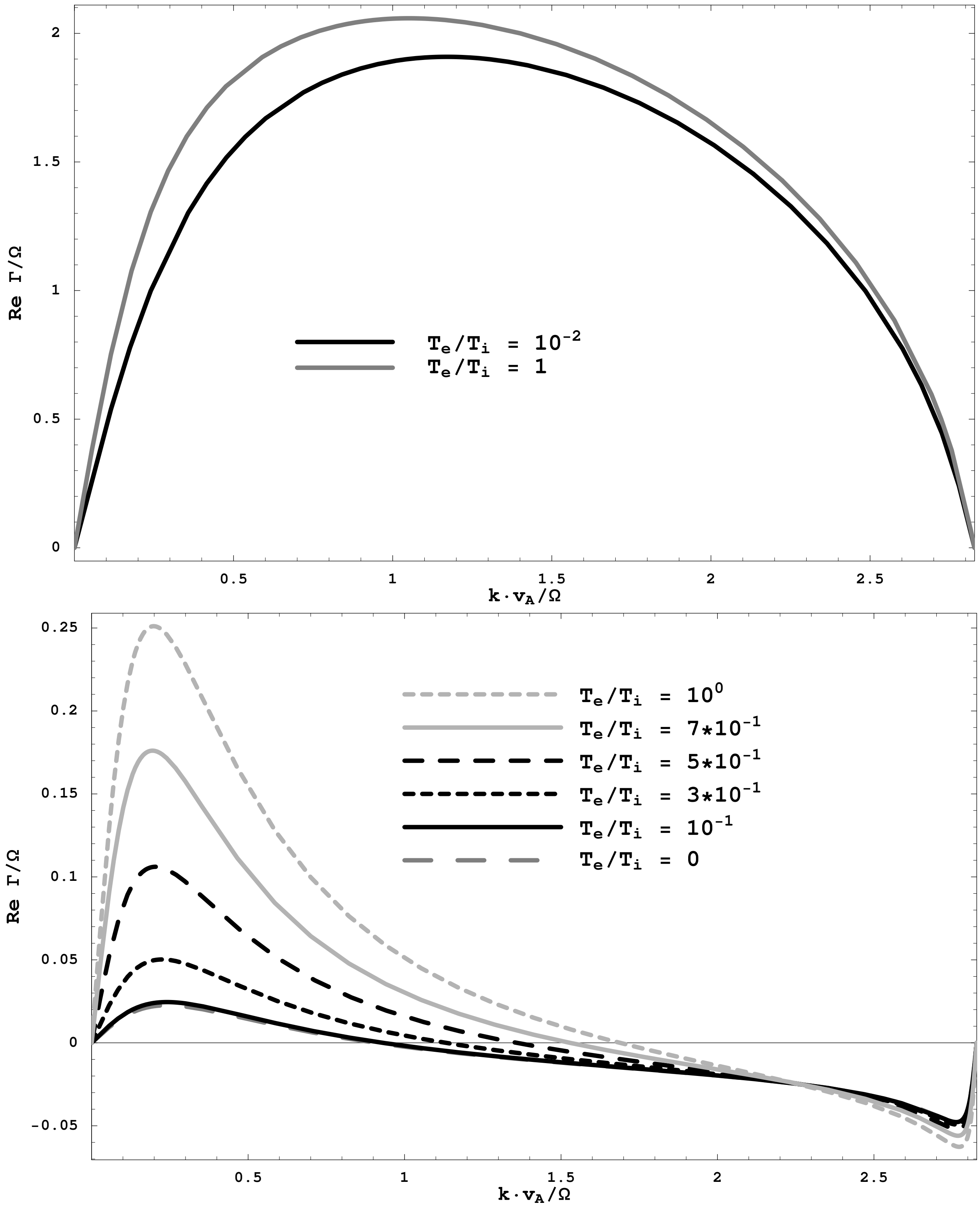

Fig. 9.— Plot of the real (top) and imaginary (bottom) parts of growth rate of the collisionless MTI. On the left we compare the real part of the growth rate for negligible and comparable electron and ion temperatures. On the right, we demonstrate that for $T_0^e/T_0^i = 10^{-1}$ the imaginary part of the growth rates coincide extremely closely with the limit of $T_0^e/T_0^i = 0$. Maximal compressible effects are reached at those wavenumbers $k_\parallel = \Omega/\theta^{1/2}$.

The discrepancy between a more careful analysis and one in which we consider only the ion dynamics is unsurprising; the phase velocity of these modes are at best of order the ion sound speed and for the bulk of unstable wavenumbers are at best of order the Alfvén speed. Effects peculiar to large electric fields and electron dynamics, remarkably, become important only in the limit that electron temperature is substantially larger than the ion temperature. Essentially, the small electron mass results in significant electric fields that lead to negligible pressure stresses relative to azimuthal stresses. This yields, for instance, the MRI dispersion relation.

B. The Forms of Lowest-Order (Quadratic) Heat and Angular Momentum Fluxes

Here we calculate the relevant perturbed quantities in expressions for the quadratic heat flux and azimuthal stress for the collisionless MRI and MTI as given in Eqs. (103) and (104). We consider only modes with vertical wavenumber, $k_R = 0$. We employ Eq. (85) with the form of the perturbed density as given in Eq. (90) and perturbed pressures as given by Eqs. (88) and (89) to determine δu_R, $\bar{B}_R$, $\bar{B}_\phi$, and δu_ϕ in terms of ξ_R. Using variable normalizations as given by Eq. (83) we have that:

$$\delta u_R = \gamma\left(\Omega \xi_R\right), \tag{B1}$$

$$\bar{B}_R = ix\left(\frac{\Omega}{v_A}\xi_R\right), \tag{B2}$$

$$\bar{B}_\phi = -\frac{2\gamma\left(b_{z0}^2 + R\left(\frac{i\gamma}{x\sqrt{2\beta}}b_{\phi 0}^2\right)\right) - ix\beta^{1/2}\alpha_P b_{\phi 0}\left[R\left(\frac{i\gamma}{x\sqrt{2\beta}}\right)-1\right]}{\gamma^2\left(b_{z0}^2 + R\left(\frac{i\gamma}{x\sqrt{2\beta}}\right)b_{\phi 0}^2\right) + x^2 - 2x^2\beta b_{\phi 0}^2\left[\left(1+\frac{\gamma^2}{2x^2\beta}\right)R\left(\frac{i\gamma}{x\sqrt{2\beta}}\right)-1\right]}\times ix\left(\frac{\Omega}{v_A}\xi_R\right), \tag{B3}$$

$$\begin{aligned}\delta u_\phi =& \frac{\gamma}{ix}v_A\bar{B}_\phi\left(b_{z0}^2 + R\left(\frac{i\gamma}{x\sqrt{2\beta}}\right)b_{\phi 0}^2\right) + \\ &\left(\Omega\xi_R\right)\left(\left|\frac{d\ln\Omega}{d\ln R}\right| - b_{\phi 0}\left[\frac{2\gamma^2}{x^2\beta}b_{\phi 0} + i\alpha_P\frac{\gamma}{x\beta^{1/2}}\right]R\left(\frac{i\gamma}{x\sqrt{2\beta}}\right)\right),\end{aligned} \tag{B4}$$

Using Eq. (46), and expressions for the perturbed heat fluxes as given in Eqs. (101) and (102), we have expressions for the δp_v, $\delta\theta$, and δq:

$$\begin{aligned}
\delta p_v = &\left(p_0^i H^{-1}\xi_R\right)\left(\left[\frac{\gamma^2}{x^2\beta}-1\right]R\left(\frac{i\gamma}{x\sqrt{2\beta}}\right)+1\right)+ \\
&2\left(p_0^i\bar{B}_\phi b_{\phi 0}-i\beta^{-1}\frac{\gamma}{x}p_0^i\xi_R\frac{\Omega}{v_A}\right)\left(\left[1+\frac{\gamma^2}{2x^2\beta}\right]R\left(\frac{i\gamma}{x\sqrt{2\beta}}\right)-1\right),
\end{aligned} \tag{B5}$$

$$\begin{aligned}
\frac{\delta\theta}{\theta} =&\frac{\delta p}{p_0^i}-\frac{\delta\rho}{\rho}=\left(\xi_R H^{-1}\right)\left(\alpha_T+\alpha_P\left(\left[\frac{5}{3}+\frac{\gamma^2}{3x^2\beta}\right]R\left(\frac{i\gamma}{x\sqrt{2\beta}}\right)-\frac{5}{3}\right)\right)+ \\
&\frac{1}{3}\bar{B}_\phi b_{\phi 0}\left(\left[\frac{\gamma^2}{2x^2\beta}-1\right]R\left(\frac{i\gamma}{x\sqrt{2\beta}}\right)+1\right)- \\
&\frac{2}{3}i\beta^{-1}\frac{\gamma}{x}b_{\phi 0}\left(\frac{\Omega}{v_A}\xi_R\right)\left(\left[1+\frac{\gamma^2}{x^2\beta}\right]R\left(\frac{i\gamma}{x\sqrt{2\beta}}\right)-1\right),
\end{aligned} \tag{B6}$$

$$\begin{aligned}
\delta q = &\left(p_0^i\Omega\xi_R\right)b_{\phi 0}\left(i\alpha_P\frac{\gamma}{x\beta^{1/2}}+\frac{2\gamma^2}{x^2\beta}\right)\left(\left[\frac{3}{2}+\frac{\gamma^2}{2x^2\beta}\right]R\left(\frac{i\gamma}{x\sqrt{2\beta}}\right)-\frac{1}{2}\right)+ \\
&i\left(p_0^i v_i\bar{B}_\phi\right)\frac{\gamma}{2x\beta^{1/2}}b_{\phi 0}\left(\left[1+\frac{\gamma^2}{x^2\beta}\right]R\left(\frac{i\gamma}{x\sqrt{2\beta}}\right)-1\right),
\end{aligned} \tag{B7}$$

Thus, with the above perturbed quantities we demonstrate the (outward) azimuthal stress and (outward) heat fluxes as shown in §4.2.