Nomenclature

$$v_A^2 = \frac{B^2}{4\pi\rho},$$

Alfvén speed

$c_s^2 \equiv$ sound speed.

$$\lambda_0 = \frac{\gamma_0}{\omega_{A_s}}$$

$$A_s = \frac{v_A^2}{c_s^2}$$

$$\omega_A = k_\| v_A,$$

shear Alfvén wave

$$\omega_{A_s} = k_\| \sqrt{\frac{v_A^2 c_s^2}{v_A^2 + c_s^2}},$$

magnetosonic wave

$$\alpha_k = \frac{2}{3}\left|\frac{d\ln\Omega}{d\ln R}\right|,$$

rot. shear

$$\alpha_z = \frac{3B_\phi}{2B_z}\alpha_k,$$

$$\delta_0 = \frac{\omega_{A_s}}{|n^0\Omega'|} = -\frac{\omega_{A_s}}{\Omega k_z \alpha_z},$$

$$r = \frac{R - R_0}{\delta_0},$$

$$K_s = \frac{\omega_{A_s}}{b_\phi \Omega}$$

$$D_T = D_\mu + \frac{D_m c_s^2}{c_s^2 + v_A^2} + \frac{D_p v_A^2}{c_s^2 + v_A^2}$$

total diffusion coeff.

Magnetorotational Instability

The magnetorotational instability (MRI)[1, 2] was reintroduced to magnetized accretion disks[3] as a fast, powerful method to transport angular momentum in accretion disks.

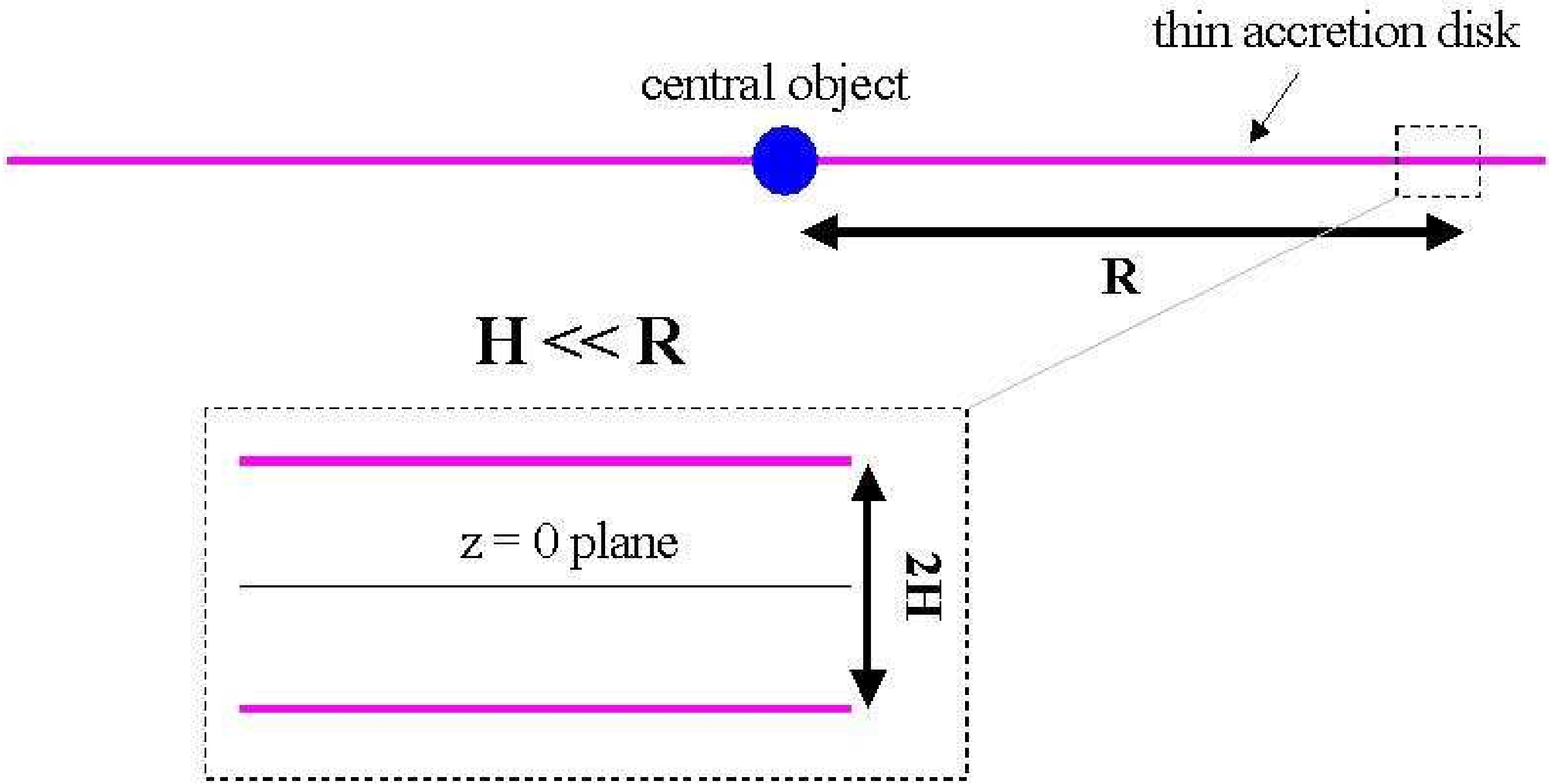
- transports particles and angular momentum in MHD plasma.
- maximum growth rate $\gamma \sim \Omega$.
- wavenumber of fastest modes $k \sim v_A/\Omega$.
- diffusion coefficient $D \equiv k^{-2}\gamma \sim \alpha_{SS} c_s H$[4], where $\alpha_{SS} = v_A^2/c_s^2 \equiv$ Shakura-Sunyaev α parameter.

Justification for 3D (Nonaxisymmetric) Modes

- The axisymmetric MRI may be approppriate for thick disks, where $c_s \sim v_\phi \ll v_{Az}$.
- For thin disks, the appropriate instabilities are ballooning modes: [5, 6, 7, 8] $\hat{v}_R \simeq \tilde{v}_R(z) \exp\left(\gamma_0 t + ik_R\left(R - R_0\right)\right)$.
  - they are characterized by $\Delta_R \ll \Delta_z \ll H$, disk height.
  - This requires that $v_A^2 \ll c_s^2$, limiting applicability.
  - Properly localized mode (i.e. mode decreases as quickly in z or quicker than density) with *discrete* eigenmodes (discrete values of k_R, γ_0); therefore, it is difficult to construct radially localized packets.

Accretion Disk Structure

- The disk is cold, hence $c_s \ll v_\phi = R\Omega$. This implies thin disk – H (disk height) $\ll R$ (disk radius): $c_s/v_\phi \sim H/R$.
- The disk is self-gravitating, $M_{\rm disk} \ll M_{\rm central\ object}$, which implies a largely Keplerian rotation profile $\left|\dfrac{d\ln\Omega}{d\ln R}\right| \simeq \dfrac{3}{2}$.
- Radial magnetic fields $B_R \ll B_z, B_\phi$.
- large ionization fraction & astrophysically *very* long mean free paths – MHD condition may be valid within much of disk.



The structure of cold accretion disk, where disk height $H \ll R$.

Properties of Instability[5]

- equilibrium conditions are given by the following:
  - magnetic field: $\mathbf{B} \simeq B_\phi \mathbf{e}_\phi + B_z \mathbf{e}_z$.
  - velocity: $\mathbf{v} = R\Omega(R)\mathbf{e}_\phi$
- Modes are three-dimensional (nonaxisymmetric):

  $$\hat{\boldsymbol{\xi}} = \tilde{\boldsymbol{\xi}}\,(R) \exp\left([\gamma_0 - i\omega]\, t + ik_z z + in^0\phi\right)$$

  Where $\boldsymbol{\xi} \equiv \dfrac{d\mathbf{v}}{dt}$ is the vector displacement, n^0 is the azimuthal wavenumber, and$\gamma_0 > 0$ is the growth rate.
- Finite compressibility, $\nabla \cdot \hat{\mathbf{v}} \neq \mathbf{0}$, also implies a finite plasma pressure $\hat{p} \neq 0$.
- Modes are radially localized about corotation point, so that frequency $\omega = n^0\Omega$.
- $\left|k_\parallel\right| = |(\mathbf{k} \cdot \mathbf{B})\,/B| \ll |k_z|$, allowing for large range of wavenumber $\mathbf{k}$ and large range of magnetic field ratios α_z that may excite this instability.

The Master Equation[6]

- $$\frac{-\alpha_z^2\left(\left[(r-r_b)^2-2i\lambda_0(r-r_b)+\frac{2r_b\delta_1}{Y(\{r-r_b\}/\delta_1;\lambda_0/\delta_1)}\right]\frac{d\tilde{\xi}}{dr}\right)}{K_s^2\left(r_b^2-r^2+2i\lambda_0 r\right)\tilde{\xi}+\frac{3\alpha_k+(4-3\alpha_k)\left(r^2-2i\lambda_0 r\right)}{(r-1)^2-2i\lambda_0(r-1)+\frac{2\delta_1}{Z(\{r-1\}/\delta_1;\lambda_0}}}$$

- $(x - i\Gamma_0 + i\mathcal{O}_{\text{even}})\, Z(x;\Gamma_0) = 1$
- $\left(x - i\Gamma_0 + i\frac{D_p+D_m}{D_\mu+D_m c_s^2/(v_A^2+c_s^2)+D_p v_A^2/(c_s^2+v_A^2)}\frac{d^2}{dx^2}\right) Y(x;\Gamma_0) = 1.$

- $\delta_1 = \left(D_T/\left(\omega_{A_s}\delta_0^2\right)\right)^{1/3}$ $\qquad r_b = \sqrt{1+A_s}$

Where $\mathcal{O}_{\text{even}}$ is some even operator (i.e. diffusion, dissipation) that smooths out singularities at $r^2 = 1, 1 + A_s$.

Mode Properties[5]

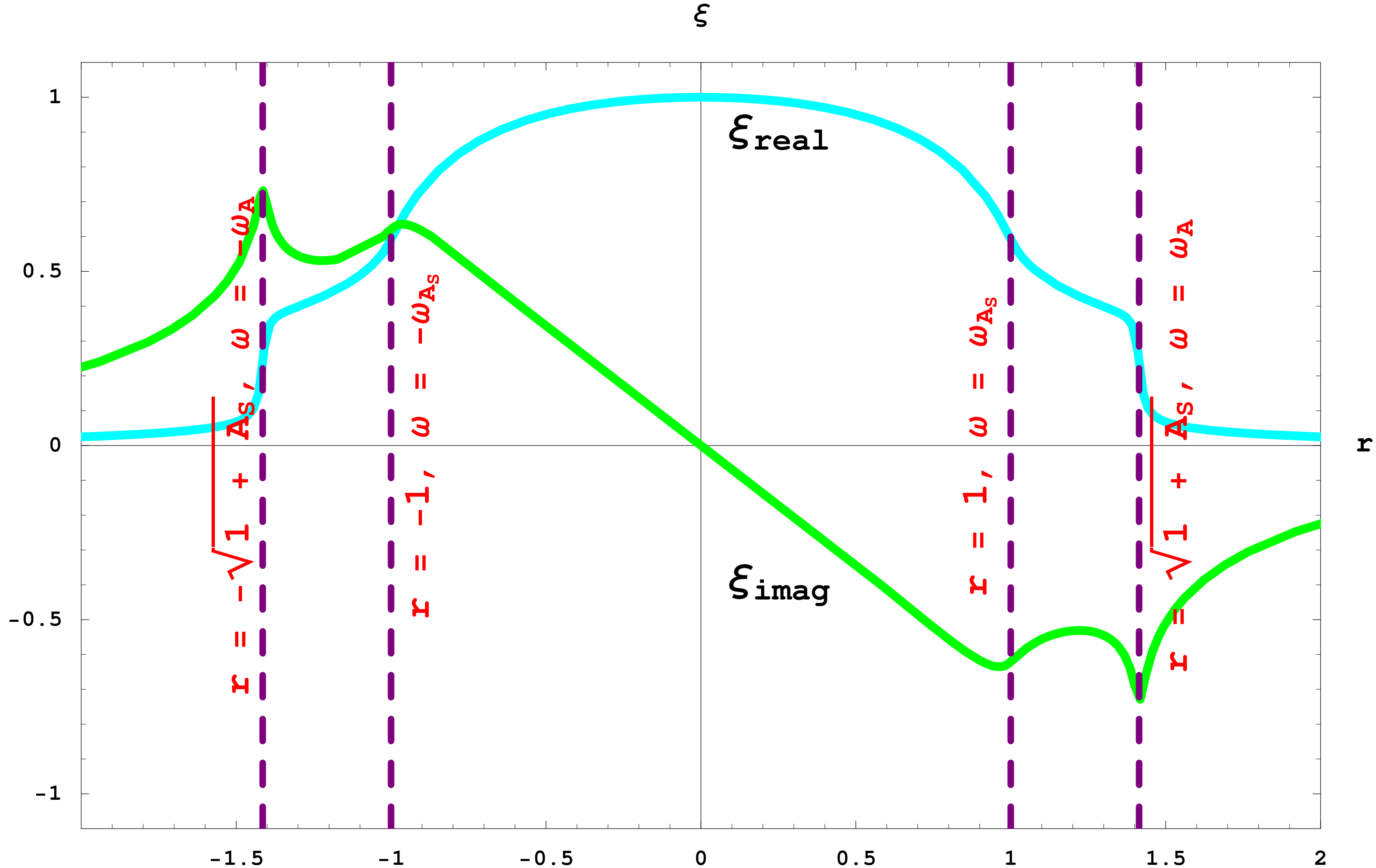
- Eigenmodes with eigenvalues K_s are found for which $\mathrm{Re}\,\tilde{\xi}$ is even, $\mathrm{Im}\,\tilde{\xi}$ is odd.

- These modes are excited for relatively large magnetic energy densities, $v_A^2 \sim c_s^2$.

- Mode is localized over distances $\delta_0 = \omega_{A_s} / \left(\Omega k_z \alpha_z\right) \sim k_{\|} H / k_z \ll R$.

- Small growth rates given by following:
$\gamma_0 \sim \omega_{A_s}^{2/3} \left(D_T \delta_0^{-2}\right)^{1/3} = \left(D_T k_z^2 \Omega^2 \alpha_z^2 / 2\right)^{1/3} \ll \omega_{A_s}$.

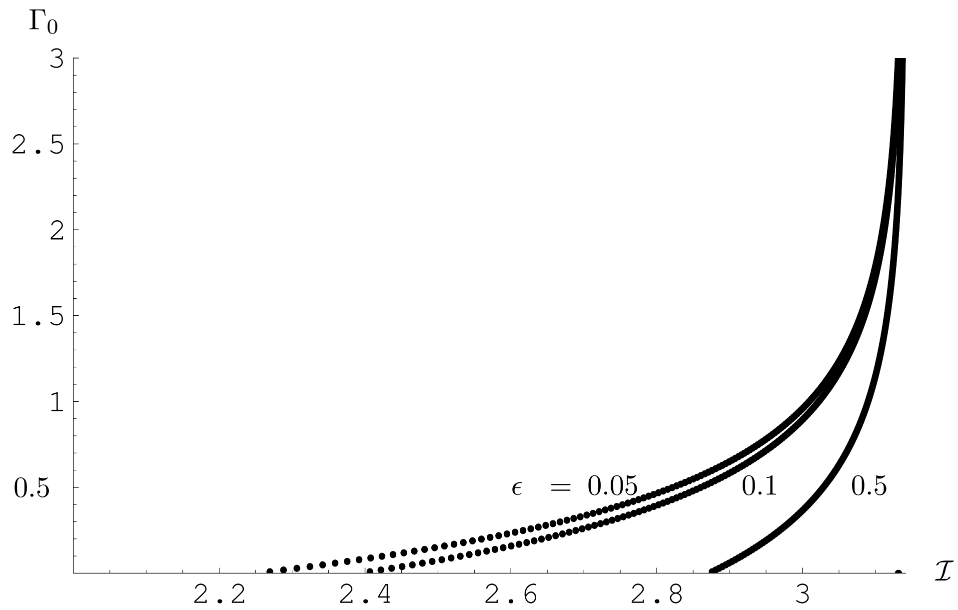
- The width of the transition layers: $\Delta_1 = \left(D_T / \left(2 k_z \Omega \alpha_z\right)\right)^{1/3} \ll \delta_0$.



- In the MHD approximation to the mode, singularities appear at radii $R - R_0 = \pm\delta_0$ (doppler shift $\bar{\omega} = \pm\omega_{A_s}$) and at $R - R_0 = \pm\delta_0\sqrt{1 + A_s}$ (doppler shift $\omega = \pm\omega_A$).

Transition Regions [5, 9]

- The characteristic function that is used to determine range of normalized growth rates $\Gamma_0 = \lambda_0/\delta_1$:
$$\mathcal{I}\left(\Gamma_0\right) = \int_{-\infty}^{\infty} \operatorname{Im} Z\left(x;\Gamma_0\right)\, dx \propto \frac{1}{\operatorname{Im}\xi(1)} \left(\frac{d\tilde{\operatorname{Im}}\,\xi}{dr}\bigg|_{r=1^+} - \frac{d\tilde{\operatorname{Im}}\,\xi}{dr}\bigg|_{r=$$
- Once can introduce a varying operator $\mathcal{O}_{\text{even}}(x)$ such that $\mathcal{I}\left(\Gamma_0\right) \neq \pi$, allowing for instability range $0 < \Gamma_0 < \Gamma_0^{\max}$.

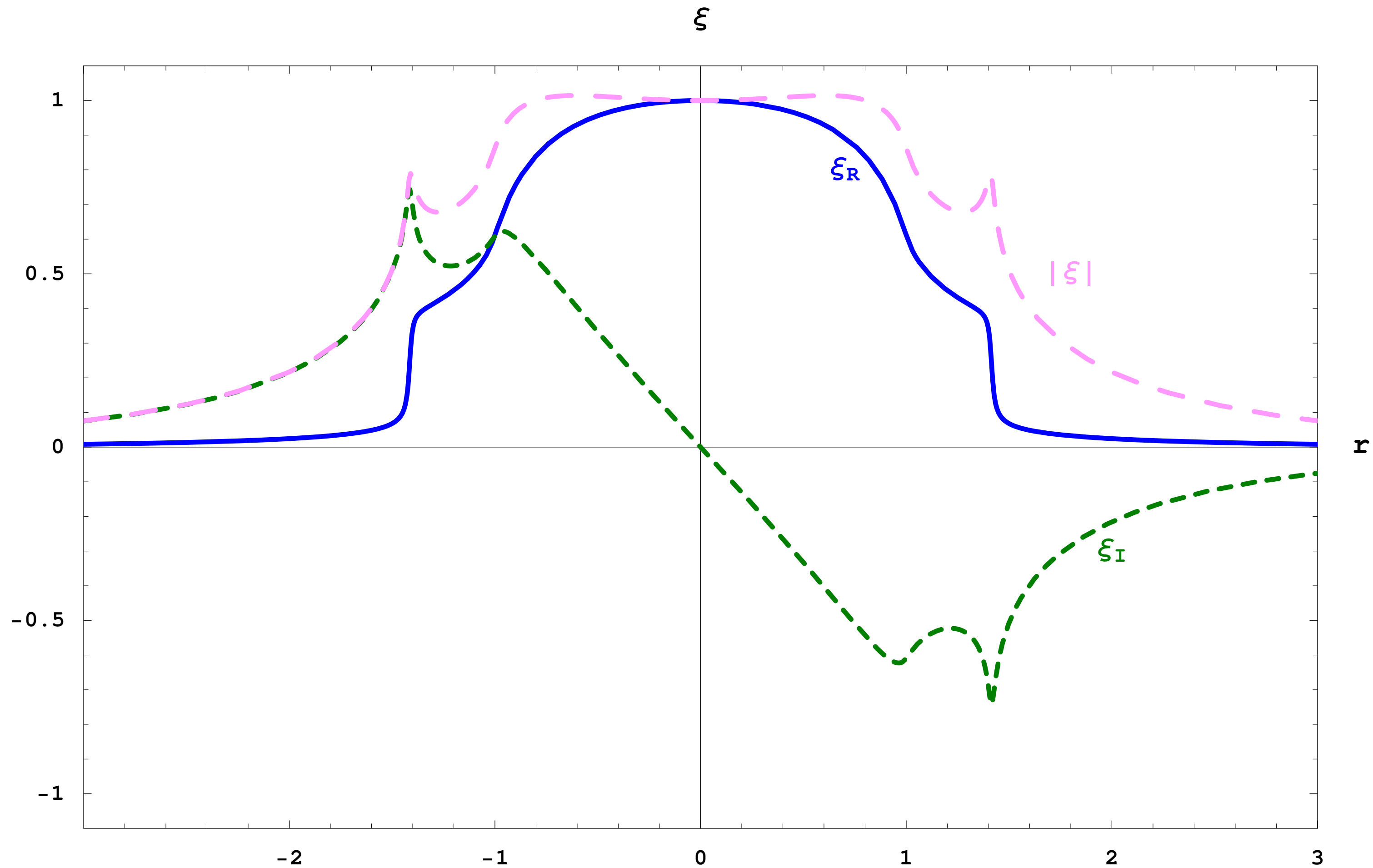


plot showing relation $\mathcal{I}$ using a representative model of the transition.

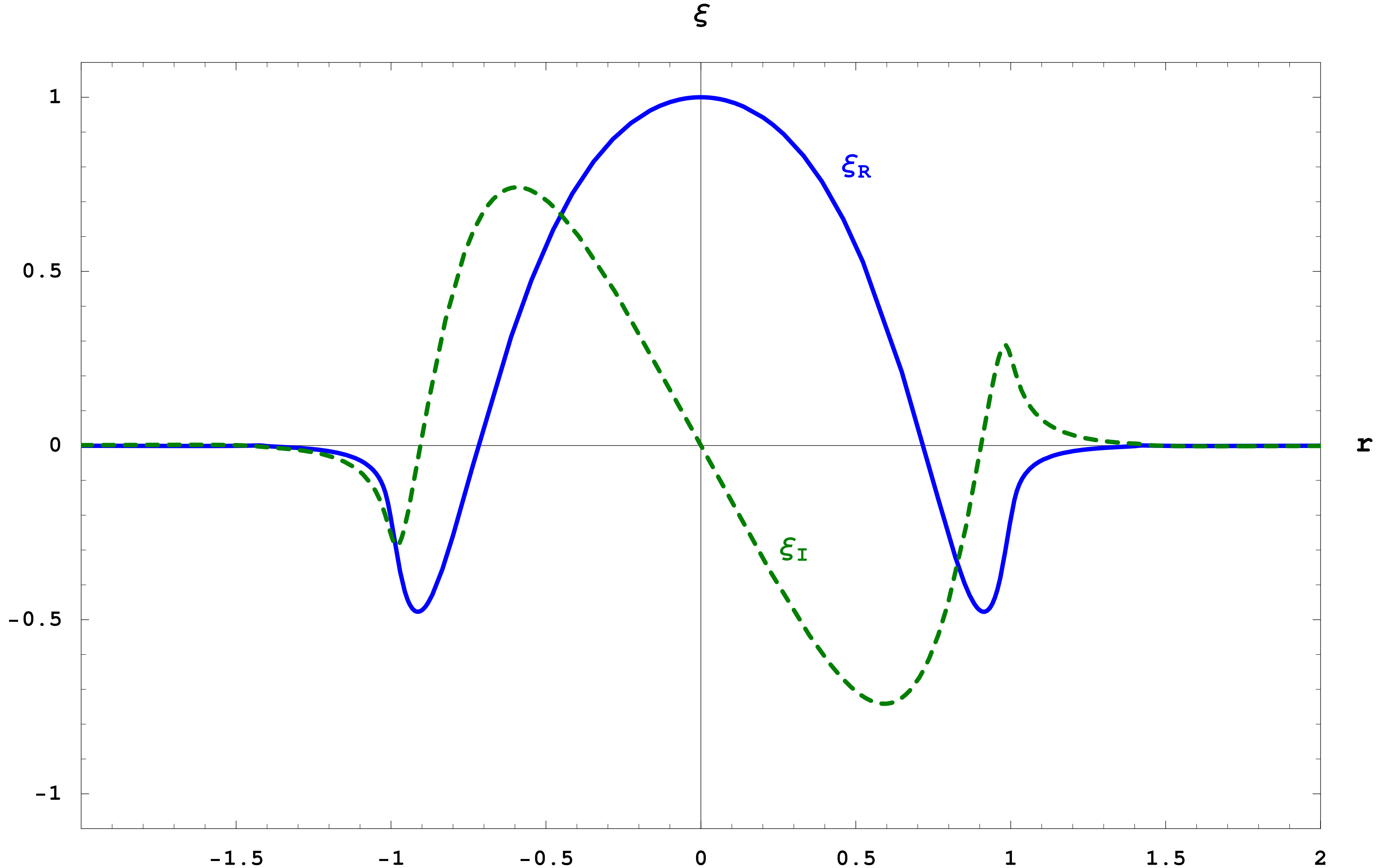
- Previous work on the form of function Z across transition region employed following models[6, 7]:
  - simplest: $\left(x - i\Gamma_0 - \frac{i\nu_{0L}}{x^2+\epsilon_a^2}\right) Z\left(x;\Gamma_0\right) = 1.$
  - inclusive: $\left(x - i\Gamma_0 - \frac{i\nu_{0L}}{x^2+\epsilon_a^2} + i\frac{d^2}{dx^2}\right) Z = 1.$
  - diffusive: $\left(x - i\Gamma_0\right) Z + i\frac{d}{dx}\left(\frac{\epsilon + x^2}{1+x^2}\frac{dZ}{dx}\right) = 1.$
  - nonlinear: $\left(x - i\Gamma_0\right) Z + |Z|^2 = 1.$

  These preserve parity across transition ($\mathrm{Im}\, Z$ is even, $\mathrm{Re}\, Z$ is odd), and $\lim_{|x|\to\infty} Z\left(x;\Gamma_0\right) \to \left(x - i\Gamma_0\right)^{-1}$.
- For linear transition operators, $\delta_1 = \left(D_T/\left(\delta_0^2\omega_{A_s}\right)\right)^{1/3}$.

General Mode Structure

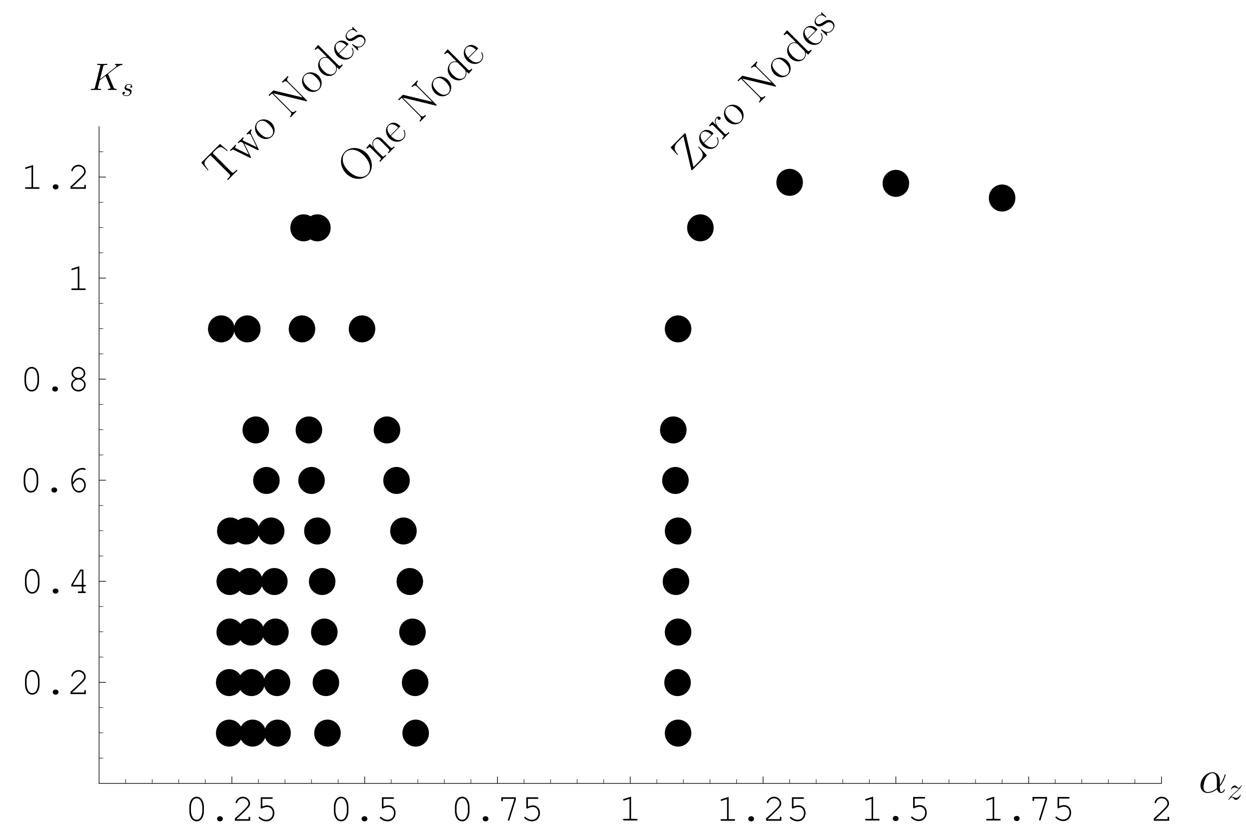


- Zero node solution for $\alpha_z = 1.6$, $A_s = 1$, $\lambda_0 = \delta_1 = 0.01$, $\epsilon_a = 0.1$, with eigenvalue $K_s = 0.887$, for simplest transition model.



• Zero node solution for $\alpha_z = 1.6$, $A_s = 1$, $\lambda_0 = \delta_1 = 0.01$, $\epsilon_a = 0.1$, with eigenvalue $K_s = 1.026$, for simplest transition model.

- Near marginal stability $\lambda_0 \rightarrow 0$, the modal spectrum is largely independent of details of the (parity-preserving) transition model[6].



Mode spectrum for simplest transition model, $\lambda_0 = 0.001$, $\delta_1 = 0.01$, $A_s = 1$, $\nu_{0L} = 1$, $\epsilon_a = 0.1$.

Nonlinear Transition Model[7]

- From physical arguments, since $\tilde{\zeta}$ (normalized velocity parallel to magnetic field) $\rightarrow \infty$, the transition function Z:

$$\left(x - i\Gamma_0\right) Z + \left(\frac{\left|\tilde{\xi}(r=1)\right|}{L_N \delta_1^2} \right) |Z|^2 + i\frac{d^2 Z}{dx^2} = 1$$

  Where L_N is some length scale.

- In the limit $\left|\tilde{\xi}(r=1)\right| > L_N \delta_1^2$, the diffusion term is negligible:
  - transition layer thickness is given by $\delta_1 = \left(\left|\xi\tilde{(1)}/L_N\right|\right)^{1/2}$.
  - The transition equation reduces to: $\left(\bar{x} - i\Gamma_0\right) Z + |Z|^2 = 1$.
  - growth rate is given by $\gamma_0 \sim \Omega B_\phi / B \left(\left|\tilde{\xi}(1)\right| / L_N\right)^{1/2}$.

Quasilinear Fluxes[5, 6, 7, 9]

- Quasilinear angular momentum flux is given by:

$$\Gamma_J^{QL} \simeq R_0 \left\langle \left\langle \rho \hat{v}_R \hat{v}_\phi - \frac{\hat{B}_R \hat{B}_\phi}{4\pi} \right\rangle \right\rangle + R_0^2 \Omega \langle \langle \hat{\rho} \hat{v}_\phi \rangle \rangle$$

Quasilinear particle flux is given by:

$$\Gamma_p^{QL} \equiv \langle \langle \hat{\rho} \hat{v}_\phi \rangle \rangle$$

Where:

$$\langle \langle \ldots \rangle \rangle = \int_{-\infty}^{\infty} dr \int_{-1}^{1} \frac{dz}{H} \int_0^{2\pi} d\phi$$

- From symmetries of the master equation:
  - From[6]:
    $\frac{1}{2}\left(\hat{n}^k\hat{v}_R^{k*}+\hat{n}^{k*}\hat{v}_R^k\right)\propto r\left|v_R^k\right|^2$, (odd function)
    * This implies that particle flux $\to 0$ in quasilinear approximation.
    * These modes fall into the class of "particle mixing" modes[10].
  - The angular momentum flux is given by[7, 5, 6]:
    $$\Gamma_J^{QL,k}\simeq\gamma_0\Omega\rho R\left(\int_{-\infty}^{\infty}\left[3\alpha_k+2\frac{1-r^2-2\lambda_0^2}{\left(1-r^2\right)^2+4\lambda_0^2}\right]\left|\hat{\xi}^k\right|^2 dr\right)>0$$
    * An estimation of diffusion coefficient requires the saturation of displacement $\left|\tilde{\xi}\right|\sim\delta_0$, with following diffusion coefficients D_J:
    * $D_J\sim\gamma_0\left(\frac{B_\phi^2}{BB_z}\frac{\omega_{A_s}}{k_z\Omega}\right)^2$ for $B_z\ll B_\phi$.
    * $D_J\sim\left(\frac{D_T}{\left(k_zH\right)^2}\right)^{1/3}\left(c_sH\right)^{2/3}$ for $B_z\sim B_\phi$.

Quasilinear Heat Accumulation

- Rate of heat accumulation from single mode of wavenumber $\mathbf{k}$:

$$S_{\mathbf{k}} \simeq \left\langle\left\langle \hat{\rho} c_s^2 \nabla \cdot \hat{\mathbf{v}} - \frac{4}{3} (\Omega + \Omega' R) \frac{\hat{B}_R \hat{B}_\phi}{4\pi} \right\rangle\right\rangle_{\mathbf{k}}$$

$$\simeq \frac{16}{3} \rho \Omega^2 \gamma_0 \left(\int_{-\infty}^{\infty} \frac{1 - \frac{3}{2}\alpha_k + \left(r^2 + \lambda_0^2\right) \left(1 + \frac{3}{2}\left[\frac{A_s}{1+A_s} - b_\phi^2 \alpha_k\right]\right)}{\lambda_0^4 + 2\lambda_0^2 \left(r^2 + 1\right) + \left(r^2 - 1\right)^2} \left|\hat{\xi}^k\right|^2 dr \right)$$

- For $B_z \sim B_\phi$:

$$|S_{\mathbf{k}}| \sim \left(\frac{D_T}{c_s H}\right)^{1/3} (k_z H)^{-4/3} p\Omega$$

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