Homework #8

1. Consider the area/volume problem, and why, it is believed, that life would move away from planets and onto planetesimals. Take the earth, for instance. It has a radius of 6×10^6 meters.

   (a) What is the surface area and volume enclosed by Earth?
   Using the area and volume formulas for a sphere:

   $$V = \frac{4}{3}\pi \left(6 \times 10^6\right)^3 = 9.05 \times 10^{20} \text{ m}^3$$
   $$A = 4\pi \left(6 \times 10^6\right)^2 = 4.52 \times 10^{14} \text{ m}^2$$

   (b) Now suppose you blew up the earth into 1 trillion 10^{12} pieces, each piece 10^{-12} of Earth's volume. What is the volume of each of these pieces? *Note, the total volume of Earth material, if Earth was exploded, would be substantially larger than the current volume of Earth – a big chunk of Earth's mass is compressed by millions of atmospheres of pressure.*
   Each element has a volume of 9.05×10^8 m^3.

   (c) Now suppose each of these pieces is a sphere. Using the formula for a sphere, $\frac{4}{3}\pi R^3 = V$, where R is the radius and V is the volume, find the radius of each of these pieces.
   One can use the formula for volume to calculate the radius of each of these elements, hence $R = \left(\frac{3V}{4\pi}\right)^{1/3}$. However, one can make the approximation that since the volume is 10^{-12} that of the original, then the radius is 10^{-4} of the original, or 600 meters.

   (d) Now calculate out the *total* surface area enclosed by these elements. How does this compare to the calculated surface area of Earth – that is, what is the ratio between the total new surface area and the old surface area.
   Each element has 10^{-4} of the radius of the original Earth, so that each element has 10^{-8} of the original area. However, there are 10^{12} of these identical spheres, so the total area is 10^4 times that of the original, or 4.52×10^{18} m^2.

   (e) Suppose we wish to construct a habitat with a period of 24 hours and an acceleration of 10 m/s^2 – Earth gravity. What is the radius of rotation of this object? Remember to calculate ω for this object, and that $T = 2\pi/\omega$, where T is the rotation period and ω is the rotational frequency.
   The angular speed of rotation of this habitat:

   $$\omega = 2\pi/T = \frac{2\pi}{86400} = 7.27 \times 10^{-5} \text{ radians/second}$$

   The formula $a = \omega^2 R$ gives the acceleration. The radius of rotation:

   $$R = \frac{10}{(7.27 \times 10^{-5})^2} = 1.89 \times 10^9 \text{ m}^2$$

   This habitat would need to be over 1 million kilometers in radius.

2. Now calculate out the sort of "fundamental" shortest scales in our Universe. It is expected that these quantities depend on $G = 6.673 \times 10^{-11}$ kg^{-1} m^3 s^{-2} being the gravitational constant, $h = 6.63 \times 10^{-34}$Joule-seconds being Planck's constant, and $c = 3 \times 10^8$ m/s being the speed of light.

Using the prescription given in class, letting $G^\alpha h^\beta c^\gamma$ equal some dimensional quantity, and noting that: 1) $G \equiv M^{-1}L^3T^{-2}$, 2) $h \equiv ML^2T^{-1}$, and 3) $c \equiv LT^{-1}$:

(a) Calculate out something with units of mass $M^1L^0T^0$, and give its value.
The prescription of constructing dimensional quantities from the above parameters, $M = G^\alpha h^\beta c^\gamma$, or that:

$$\left(M^{-1}L^3T^{-2}\right)^\alpha \left(ML^2T^{-1}\right)^\beta \left(LT^{-1}\right)^\gamma = ML^0T^0$$
$$M^{-\alpha+\beta}L^{3\alpha+2\beta+\gamma}T^{-2\alpha-\beta-\gamma} = M^1L^0T^0$$

So the equations to be solved:

$$-\alpha + \beta = 1$$
$$3\alpha + 2\beta + \gamma = 0$$
$$-2\alpha - \beta - \gamma = 0$$

Solving the above equation yields $\alpha = -1/2$, $\beta = 1/2$, $\gamma = 1/2$. Therefore:

$$m_{\text{Planck}} = \sqrt{\frac{hc}{G}} = 5.46 \times 10^{-8} \text{ kg}$$

(b) Calculate out something with units of length $M^0L^1T^0$, and give its value.
With the above prescription:

$$-\alpha + \beta = 0$$
$$3\alpha + 2\beta + \gamma = 1$$
$$-2\alpha - \beta - \gamma = 0$$

So $\alpha = 1/2$, $\beta = 1/2$, $\gamma = -3/2$. Therefore:

$$\ell_{\text{Planck}} = \sqrt{\frac{Gh}{c^3}} = 4.05 \times 10^{-35} \text{ m}$$

(c) Calculate out something with units of time $M^0L^0T^1$, and give its value.
With the above prescription:

$$-\alpha + \beta = 0$$
$$3\alpha + 2\beta + \gamma = 0$$
$$-2\alpha - \beta - \gamma = 1$$

So $\alpha = 1/2$, $\beta = 1/2$, $\gamma = -5/2$. Therefore:

$$t_{\text{Planck}} = \sqrt{\frac{Gh}{c^5}} = 1.35 \times 10^{-43} \text{ s}$$

3. Now consider the habitability of a Dyson sphere.

(a) For a Dyson sphere of radius equal to 1 AU (1.5×10^{11} meters), what level of population can it support, assuming that Earth, with a radius of 6×10^6 meters, can support 10 billion people? To solve this problem, you need to find the surface areas of Earth and the Dyson sphere.

The habitable population of the Dyson sphere, assuming 10 billion people could be supported on Earth, is:

$$N_{\text{Dyson}} = \left(\frac{1.5 \times 10^{11}}{6 \times 10^6}\right)^2 \times 10^{10} = 6.25 \times 10^{18} \text{ people}$$

(b) The volume of a Dyson sphere is approximately $V = 4\pi R^2 \Delta R$, where ΔR is the thickness of the shell. If a Dyson sphere has radius of 1 AU and a thickness of 30 centimeters.

i. What is its volume?

Thickness is 0.3 m. Radius $R = 1.5 \times 10^{11}$ m. Then the volume of the Dyson sphere:

$$V = 4\pi \left(1.5 \times 10^{11}\right)^2 \times 0.3 = 8.48 \times 10^{22} \text{ m}^3$$

ii. Assume the sphere has a mass of Jupiter, so that $M_{\text{Dyson}} = 1.96 \times 10^{27}$ kg, what is the *mass density* of the Dyson sphere? How does this compare to water – what is the ratio of the Dyson sphere mass density to that that of water, with density 1000 kg/m^3?

The density of the Dyson sphere material:

$$\rho = \frac{1.96 \times 10^{27}}{8.48 \times 10^{22}} = 2.31 \times 10^4 \text{ kg/m}^3$$

This is 23.1 times that of the density of water.

4. **Extra Credit II – 36 points** The emission of radiation from any body is a function its mass M, radius R, and Planck's constant $h = 6.63 \times 10^{-34}$ J-seconds.

(a) Construct something with units of power (units of ML^2T^{-1}) from M, R, and h.

Using the prescription for dimensional formulae, we have that $P = M^\alpha R^\beta h^\gamma$. Note that h has units of Joule-seconds, so that $h \equiv ML^2T^{-1}$. The system of equations to solve:

$$ML^2T^{-3} = M^\alpha L^\beta \left(ML^2T^{-1}\right)^\gamma$$
$$ML^2T^{-3} = M^{\alpha+\gamma} L^{\beta+2\gamma} T^{-\gamma}$$

The three sets of equations:

$$\alpha + \gamma = 1$$
$$\beta + 2\gamma = 2$$
$$-\gamma = -3$$

Which has solution $\alpha = -2$, $\beta = -4$, and $\gamma = 3$, or $P = M^{-2}R^{-4}h^3$.

(b) The radius of a black hole with mass M is approximately $R \sim GM/c^2$. Now what is the power that leaves its black hole as a function of black hole mass?

Substituting in $R = GMc^{-2}$ into the above power equation, we get:

$$P = M^{-2}\left(GMc^{-2}\right)^{-4} h^3 = G^{-4}M^{-6}c^8h^3$$

Alternatively, one could set $M = Rc^2G^{-1}$ to get:

$$P = \left(Rc^2G^{-1}\right)^{-2} R^{-4}h^3 = R^{-6}c^{-4}G^2h^3$$

(c) Now given that the power output corresponds to the mass loss rate, how would I calculate the lifetime τ of the black hole, assuming that $P = Mc^2/\tau$, where P is power?
Using the formula for power emitted by a black hole, one gets that:

$$Mc^2\tau^{-1} = G^{-4}M^{-6}c^8h^3$$
$$\tau = G^4M^7c^{-6}h^{-3} = G^4\left(Rc^2G^{-1}\right)^7c^{-6}h^{-3} = R^7c^8G^{-3}h^{-3}$$

If one wished to approach the problem in terms of the black hole radius.

(d) What is the lifetime, using the above formula, for a 1 solar mass black hole, $M = 2 \times 10^{30}$ kg? How about for a 10^9 solar mass black hole?
Using the formula for the lifetime of the black hole, for a one solar mass black hole $M = 2 \times 10^{30}$ kg:

$$T = \frac{\left(6.673 \times 10^{-11}\right)^4\left(2 \times 10^{30}\right)^7}{\left(3 \times 10^8\right)^6\left(6.626 \times 10^{-34}\right)^3} = 1.20 \times 10^{220}\ \text{s} = 3.79 \times 10^{212}\ \text{yr}$$

For a 10^9 solar mass black hole, the lifetime is $10^{9\times7} = 10^{63}$ times longer, therefore $T = 1.20 \times 10^{283}$ s $= 3.79 \times 10^{275}$ yr.