Homework #2

1. The sun has mass $M_\odot = 2 \times 10^{30}$ kg, and it has a total power output of $L_\odot = 4 \times 10^{26}$ W. In its lifetime on the *main sequence* (defined as that period of time when a star burns only hydrogen within the core, rather than in a shell surrounding the core), the sun will "burn" 15% of its hydrogen. The sun initially was made up of 70% hydrogen by mass. For each 1 gram of hydrogen burned, 10^{-5} grams of energy are removed.

   (a) Using the relation $E = mc^2$, where E is the energy in Joules, m is the mass in kilograms, and $c = 3 \times 10^8$ m/s is the speed of light, how many *Joules* of energy are released when 1 *kilogram* of hydrogen is burned?
   This is a *proportionality* problem. For every 1 kg of hydrogen "burned" through nuclear fusion, 0.01 kg of energy are created. Using $E = mc^2$ formula, where $c = 3 \times 10^8$ m/s, we can calculate out the energy liberated through 1 kg of hydrogen fusing into its final products.

   $$E = (0.01 \text{ kg}) \times \left(3 \times 10^8 \text{ m/s}\right)^2 = 9 \times 10^{14} \text{ J}$$

   So 9×10^{14} J of energy are released. Remember to check that your dimensions are correct. $1 \text{ J} = 1 \text{ kg m}^2/\text{s}^2$.

   (b) Based on this, how much energy is available from fusion?
   First calculate the amount of hydrogen available in the sun for fusion burning. This amount $M_{\text{fusion}} = 2 \times 10^{30} \times 0.7 \times 0.15 = 2.1 \times 10^{29}$ kg. Given the result from (1a), the amount of total energy available becomes:

   $$E_{\text{tot}} = 2.1 \times 10^{29} \times \left(9 \times 10^{14}\right) = 1.89 \times 10^{44} \text{ J}$$

   Just for comparison, this is several orders of magnitude larger than the energy liberated from the gravitational collapse of the initial molecular cloud.

   (c) Given the sun's luminosity (which is given in Watts $\equiv$ Joules/second), what is the lifetime of the sun?
   The lifetime of the sun is given by the following, Lifetime$\equiv \dfrac{\text{energy available}}{\text{power output}}$. Thus, the lifetime of the sun is given by:

   $$T = \frac{E_{\text{tot}}}{L_\odot} = \frac{1.89 \times 10^{44}}{4 \times 10^{26}} = 4.73 \times 10^{17} \text{ s} = 1.50 \times 10^{10} \text{ yr}$$

   One of the earliest bits of evidence for another energy source fir for the sun was the fact that, given the sun's power output, the lifetime according to the energy liberated from gravitational collapse of the initial cloud would give a lifetime of only a few million years (the so-called Kelvin-Helmholtz timescale). But based on geological evidence, people knew that the earth was billions of years old.

2. Here we return to the proverbial "flies in a box" problems of dimensional analysis with length, volumes, and densities.

   (a) The average density of stars in our solar neighborhood is 0.1 pc^{-3} (0.1 stars per cubic parsec). What is the average separation between stars, in our solar neighborhood? The closest star system to ours is α Centauri, located 1.3 pc away. Is it *closer* or *farther* than the average distance?
   Recall that one is given a *volume number density* ρ, which here has units of pc^{-3} To get a length ℓ, this length being the average separation between objects, is given by $\ell = \rho^{-1/3}$, because objects here are spread out over three dimensions. Given the above density, the average stellar separation:

   $$\ell = \left(0.1 \text{ pc}^{-3}\right)^{-1/3} = 2.15 \text{ pc}$$

   α Centauri is 1.3 pc away, which is **closer** than the average star-star distance.

   (b) The 50 largest metropolitan areas in the United States have populations larger than 1 million people, with a total area of 9 million km^{-2}. If these metropolises were distributed evenly across its land mass, and given what you know about areal density, what would be the average separation between 2 metropolises?
   Here objects are distributed over an area (2D) rather than a volume (3D). We have an *areal number density* σ, which has units of km^{-2}. To get a length, we take $\ell = \sigma^{-1/2}$. First, calculate the areal number density:

   $$\sigma = \frac{50}{9 \times 10^6 \text{ km}^2} = 5.56 \times 10^{-6} \text{ km}^{-2}$$

   And the average separation between cities becomes:

   $$\ell = \left(5.56 \times 10^{-6} \text{ km}^{-2}\right)^{-1/2} = 424 \text{ km}$$

Of course, these metropolises are not distributed in this fashion. Almost all are concentrated on the East Coast, West Coast, North-Central US (Chicago, Illionois, Michigan, Ohio), and Texas. Just a few (5 or so) lie outside these regions. A very quick and accurate gauge of relative population density in the U. S. is to look at this country from space at night – the brighter regions have a higher population density.