Review For Test #1 With Answers

The material on this test will be a comprehensive and representative sampling of material to review on the first test, to be given on Monday, February 23. As such, there will be five main topics (General Relativity, Black Holes, Accretion and Binary Stars, and the Milky Way [With Other Galaxies]), and there will be 5 multiple choice and one essay question per main topic. Answers will be displayed in red.

General Relativity

1. You shine a flashlight parallel to the ground in your "accelerating" elevator. The elevator moves **upward** with acceleration g. The light moves a horizontal distance ℓ. According to general relativity, the light ray will bend:
   (a) $\ell^2 g/(2c)$ **downwards**.
   (b) $\ell^2 g/(2c)$ **upwards**.
   (c) $\ell^2 g/c$ **upwards**.
   (d) $\ell^2 g/c$ **downwards**.
   (e) $\ell^2 g/c$ **backwards** (towards the direction of the incoming flashlight beam).

2. A neutron star has mass 1 $M_\odot$ and radius 10 km. As seen by a distant observer at Earth, a flare on its surface is observed to last 1 microsecond. To an observer at the surface, the flare lasts:
   (a) 1.2 microseconds.
   (b) 0.84 microseconds.
   (c) 0.92 microseconds.
   (d) 1.1 microseconds.

3. Mach's principle states implies that:
   (a) A person accelerating in a closed elevator cannot distinguish whether he is accelerating upwards or a gravitational force is pushing him downwards.
   (b) The acceleration on matter due to gravity is independent of mass.
   (c) A person spinning in a universe filled with stationary matter will feel the same force as if he were stationary and all the matter were spinning around him.
   (d) The path that light travels in a gravitational field is curved.

4. The mass-energy density of the universe is approximately 10^{-27} gm cm^{-3}. From purely dimensional arguments, the distance scale over which general relativistic effects become dominant is $R^2 \sim c^2/(G\rho)$. This length scale is approximately on the order of:
   (a) 10 million parsecs.
   (b) 100 million parsecs.
   (c) 1 billion parsecs.

(d) 10 billion parsecs.

5. One twin travels at 99% of lightspeed to α Centauri and back, while the other remains at Earth. α Centauri is 4.5 light-years away. The difference in their ages when the twins again meet is:

   (a) 0 years.

   (b) 7.8 years.

   (c) 3.9 years.

   (d) 9 years.

LIGO Detection of Gravitational Waves. The LIGO detection system can detect a gravitational wave signal of strain as small as 10^{-18}. For example, a strain of 10^{-4} (seen with bone and othe rigid materials) on a femur, of length 30 cm, means that the length will change by $30 \times 10^{-4} = 3 \times 10^{-3}$ cm.

1. The LIGO detector works by shining a laser beam and measuring a *phase shift* between incoming and bounced light – bounced between two very highly reflective mirrors. A typical wavelength of the light is 5×10^{-5} cm. If the strain is 10^{-18}, how far does the light have to travel to get a path length change of 10^{-3} of a wavelength (the minimum for detection)?
   The strain on the path of length D is such that the change $\Delta D = hD = 10^{-2}\lambda$. Given the value of $\lambda = 5 \times 10^{-5}$ cm, then:

$$D = h^{-1} \times 10^{-2} \times \left(5 \times 10^{-5}\right) \text{ cm}$$
$$D = 5 \times 10^{10} \text{ cm}$$

2. The bounce length of light between the two LIGO mirrors is 2 kilometers. How many bounces are required to get a measurable amount of strain – you may consider 2 km as one "bounce."
   The number of bounces $Nd = D$, where $d = 2 \times 10^5$ cm. The number of bounces $N = 5 \times 10^{10}/\left(2 \times 10^5\right) = 2.5 \times 10^5$ bounces.

3. **bonus**. At each bounce The laser loses only 10^{-5} of the power before the bounce. If a detection requires that the final bounce have power higher than 10^{-6} of the initial power, is the detection "unambiguous"?
   At each bounce one loses beam power of 10^{-4}. That is:

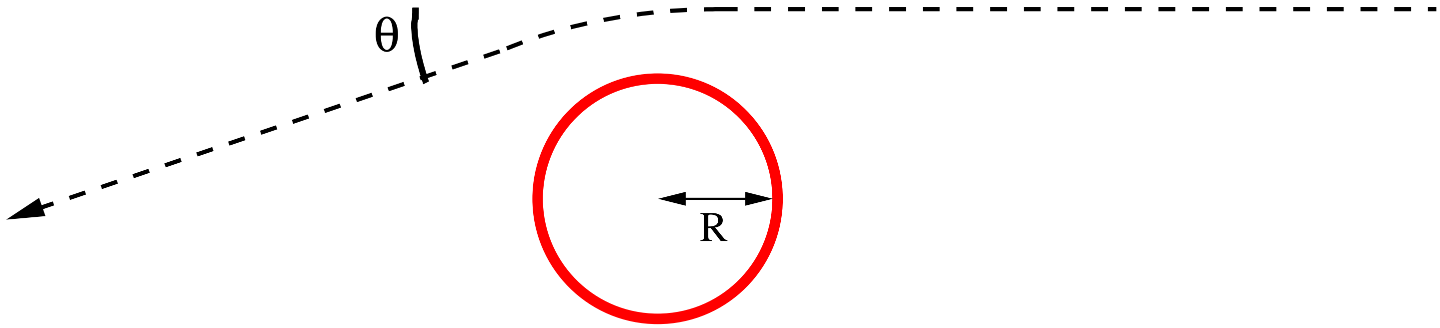
$$P_{n+1}/P_n = 1 - 10^{-4}$$

   Which implies that $P_n/P_0 = \left(1 - 10^{-4}\right)^n$. After $n = 250,000$ bounces, taking the natural logarithm of both sides:

$$\ln\left(P_n/P_0\right) = n \ln\left(1 - 10^{-4}\right) = -2.5$$
$$P_n = 8.2 \times 10^{-2} P_0$$

   So the pulse remains visible, allowing for the measurement of such a small gravitational wave strain.

Small-Angle Bending of Starlight. Assume a star with radius much smaller than that of the sun. Show from a simple order of magnitude estimate that the bending angle of starlight that grazes a star goes as $\theta \sim GM/(Rc^2)$, with R being the radius of the star and M its mass. A diagram is shown below of this problem.



- The light travels with speed c. Calculate the time required for the light to travel a distance R. Denote this time Δt.
  The travel time for light, under which it feels the effect of this force, is $\Delta t = R/c$.

- In this time Δt, the photon acquires a transverse velocity of the order of $\delta v_\perp \sim g\Delta t$. Use g, the gravitational acceleration as calculated from a Newtonian approximation.
  Newtonian gravitational acceleration

$$g = \frac{GM}{R^2}$$

  So that the change in velocity:

$$\delta v_\perp \sim g\Delta t = \frac{GM}{R^2} \times \frac{R}{c} = \frac{GM}{Rc}$$

- The small angle $\theta \approx \delta v_\perp / c$.
  The small angle for very small changes in velocity:

$$\theta \approx \frac{\delta v_\perp}{c} \sim \frac{GM}{Rc^2}$$

  In Newtonian gravity, $\theta \approx GM/(Rc^2)$. In general relativity, $\theta \approx 2GM/(Rc^2)$.

Black Holes

1. The average density of a 10^9 solar mass black hole is approximately:

   (a) 5×10^{-2} gm cm^{-3}.

   (b) 10^3 gm cm^{-2}.

   (c) 10^6 gm cm^{-3}.

   (d) 10^{-4} gm cm^{-3}.

2. In general relativity the angular momentum a has units of length^2. Using G and c, a in terms of the physical angular momentum ℓ in term

   (a) $a = G\ell^2/c$.

   (b) $a = G\ell/c^3$.

   (c) $a = G^2\ell/c^2$.

   (d) $a = G\ell/c^3$.

3. The creation of a *stable* wormhole, that allows for the passage of matter, is believed to require:

   (a) A quickly rotating black hole.

   (b) A highly charged black hole.

   (c) A tunnel threaded with *exotic* matter (matter with negative energy density).

   (d) All black holes have stable wormholes.

4. A quantum black hole has a mass of 2×10^{14} gm. Given that the evaporation time for black holes is $t_{\rm evap} = 10^{66} \left(M/M_\odot\right)^3$ yr, its lifetime is:

   (a) 10^{20} years.

   (b) 10^{24} years.

   (c) 10^8 years.

   (d) 10^9 years.

5. The angular rotation frequency Ω at which spacetime is dragged is proportional to ℓ, the angular momentum. Using G and c and R (the radius of the spinning matter), dimensionally $\Omega \sim$:

   (a) $\Omega \sim G\ell/\left(r^2c^3\right)$.

   (b) $\Omega \sim G^2\ell/\left(rc\right)$.

   (c) $\Omega \sim G^3\ell/\left(r^4c^4\right)$.

   (d) $\Omega \sim G\ell/\left(r^3c^2\right)$.

Tidal Acceleration Falling Into Black Holes. Imagine an object of length L falling through the event horizon. The tidal acceleration (the difference in acceleration between the top and bottom of the object) goes as $a = Lg/R$ (assumed that $L \ll R$), where g is the local gravitational acceleration and R is a radial distance.

1. Given that $R_s = 2GM/c^2$. What is the gravitational acceleration g at the event horizon, using the formula $g = GM/R_s^2$?

$$g = GM\frac{c^4}{4G^2M^2} = \frac{c^4}{4GM}$$

2. What is the tidal acceleration at the event horizon?

$$a = \frac{gL}{R_s} = \frac{c^4 L}{4GM} \times \frac{c^2}{2GM} = \frac{c^6}{8G^2M^2}L$$

3. What is the tidal acceleration felt by a 2 m astronaut falling into a 10^9 $M_\odot$ black hole?
The mass of the black hole $M = 2 \times 10^{33} \times 10^9 = 2 \times 10^{42}$ gm. The tidal acceleration felt by such a person:

$$a = \frac{c^6}{8G^2M^2}L = 1.02 \times 10^{-10} \text{ cm s}^{-2}$$

Resulting in quite manageable tidal stresses.

Location of the Black Hole Static Limit From Dimensional Arguments. Consider a black hole with mass M and angular momentum L. This spinning black hole will drag spacetime around it, such that gyroscopes will precess with frequency Ω.

1. From dimensional arguments, using the angular momentum ℓ, G, c, and R (the distance from the black hole), calculate the precession frequency $\Omega = \ell G^\alpha c^\beta R^\gamma$.
The dimensions of $\Omega \equiv T^{-1}$, $G \equiv M^{-1}L^3T^{-2}$, $c \equiv LT^{-1}$, and $R \equiv L$, and $\ell \equiv ML^2T^{-1}$, where M is a dimension of mass, L of length, and T of time. Solving the equation:

$$M^0L^0T^{-1} = M^{1-\alpha}L^{2+3\alpha+\beta+\gamma}T^{-1-2\alpha-\beta}$$

The system of equations:

$$\begin{aligned}
&1 - \alpha = 0\\
&2 + 3\alpha + \beta + \gamma = 0\\
&-1 = -1 - 2\alpha - \beta
\end{aligned}$$

Solving for which $\alpha = 1$, $\beta = -2$, $\gamma = -3$. The solution is $\Omega = G\ell/(R^3c^2)$.

2. Take the value of Ω as found in the above. The location of the static limit can be found by setting $R\Omega = c$. Find the R for which all motion is forced to move with the black hole, as a function of L and G.
   Solving the equation:

$$R\Omega = c = \frac{G\ell}{R^2c^2}$$
$$R = \sqrt{\frac{G\ell}{c^3}}$$

3. The maximum value of the angular momentum of a system is $\ell_{\text{max}} = GM^2/c$. What is the radius of the static limit in this case?

$$R = \sqrt{\frac{G^2M^2}{c^4}} = \frac{GM}{c^2}$$

   Which is of the order of the size of the Schwarzchild radius of a star of mass M.

Accretion and Binary Stars

1. A white dwarf of 1 $M_\odot$ has an accretion rate of 10^{-9} $M_\odot$ yr^{-1}. The radius of the star is 6×10^8 cm. The luminosity of the accretion is approximately:
   (a) 200 $L_\odot$.
   (b) 20 $L_\odot$.
   (c) 2 $L_\odot$.
   (d) 0.2 $L_\odot$.

2. The Alfvén radius is the radius at which:
   (a) The gas pressure density equals the magnetic energy density.
   (b) The gas kinetic energy density equals the magnetic energy density.
   (c) The radius at which the speed of the particles approaches light speed.
   (d) The radius **below which** gas kinetic energy density dominates over magnetic energy density.

3. The spinning down of pulsars is due to torques caused by:
   (a) Friction with the interstellar medium.
   (b) Friction with an accretion disk.
   (c) Electromagnetic radiation from a nonaligned (rotation axis nonaligned with magnetic axis) magnetic dipole.
   (d) Tidal stresses due to an orbital companion.

4. From the virial theorem, the emission of thermal radiation in the X-rays onto an accreting black hole is due to the fact that:
   (a) The surface blackbody emission from these objects is in the X-rays, so material accreting onto the surface will be heated by the surface to emit X-rays.
   (b) Accretion is a very efficient process (on the order of a few percent of the rest mass energy), so matter will emit thermally X-ray radiation.
   (c) Their large magnetic fields produce synchrotron emission in the X rays.
   (d) The shock-like interaction between the infalling matter and the surface results in X-ray emission.

5. Assume a neutron star and a white dwarf have the same mass, but different sizes. The neutron star has a radius of 10 km while the white dwarf, one of 10000 km. If they both have the same mass accretion rate, then:
   (a) the white dwarf will be 1000 times brighter than the neutron star.
   (b) The neutron star will be 10^6 times brighter than the white dwarf.
   (c) The white dwarf will be 10^6 times brighter than the neutron star.
   (d) The neutron star will be 1000 times brighter than the white dwarf.

Spin-Up Rate of Neutron Stars. The accretion rate for matter onto a 10 km neutron star is 10^{-9} M$_\odot$ yr^{-1}. The magnetic field has strength such that the Alfvén radius is $R_A = 10^9$ cm. Spin-up is due to matter dragging along the star (with magnetic field) at the Alfvén radius.

(a) Using $v = \sqrt{2GM_{ns}/R_A}$, $\dot{M}$, and R_A, construct something that has units of rate of change of angular momentum (i.e., torque). That is $dL/dt = \dot{M}^\alpha v^\beta R_A^\gamma$.
Dimensionally, the torque (rate of change of angular momentum) $\dot{L} = \dot{M} v R_A = \dot{M}\sqrt{2GM_{ns}R_A}$.

(b) What is the torque, in gm cm^2 s^{-2}, given the above parameters?
Given the above expressions, the torque on the neutron star, since $\dot{M} = 6.34 \times 10^{13}$ gm s^{-1}.

$$\dot{L} = \dot{M}\sqrt{2GM_{ns}R_A} = 3.3 \times 10^{34} \text{ gm cm}^2 \text{ s}^{-2}$$

(c) The neutron star has moment of 8×10^{46} gm cm^2. What is the spin-up rate of the neutron star? That is, from the formula:

$$\frac{dL}{dt} = I\frac{d\omega}{dt}$$

Calculate out $d\omega/dt$.
Putting in the above values:

$$\dot{\omega} = \frac{\dot{L}}{I} = \frac{3.3 \times 10^{34}}{8 \times 10^{46}} \text{ s}^{-2} = 4.09 \times 10^{-13} \text{ s}^{-2}$$

(d) Estimate the time required for the pulsar to speed up to $P = 1$ millisecond. Recall that $P = 2\pi/\omega$.
The time required (here we do not solve a differential equation), is estimated as:

$$\dot{\omega} = \omega_F/\tau$$
$$\tau = \frac{I\omega_F}{\dot{L}}$$
$$\tau = \frac{2\pi I}{\dot{L}P} = 1.5 \times 10^{16} \text{ s} = 4.9 \times 10^8 \text{ yr}$$

Roche Envelope of Star. Imagine a 1 M$_\odot$ star with a radius of 7×10^{10} cm. Here you will answer questions about a configuration in which this star overfills its Roche lobe.

(a) What is the surface gravity, in cm s^{-2}, at the surface of this star?
The gravitational acceleration

$$g = \frac{GM}{R^2} = 2.7 \times 10^4 \text{ cm s}^{-2}$$

(b) Assume a this binary consists of a 1 $M_\odot$ star and a 1 M$_\odot$ neutron star. What do the separation of the stars from each other have to be in order for the net acceleration on the surface of the main sequence star, facing the neutron star, cancel out? The diagram is shown below.

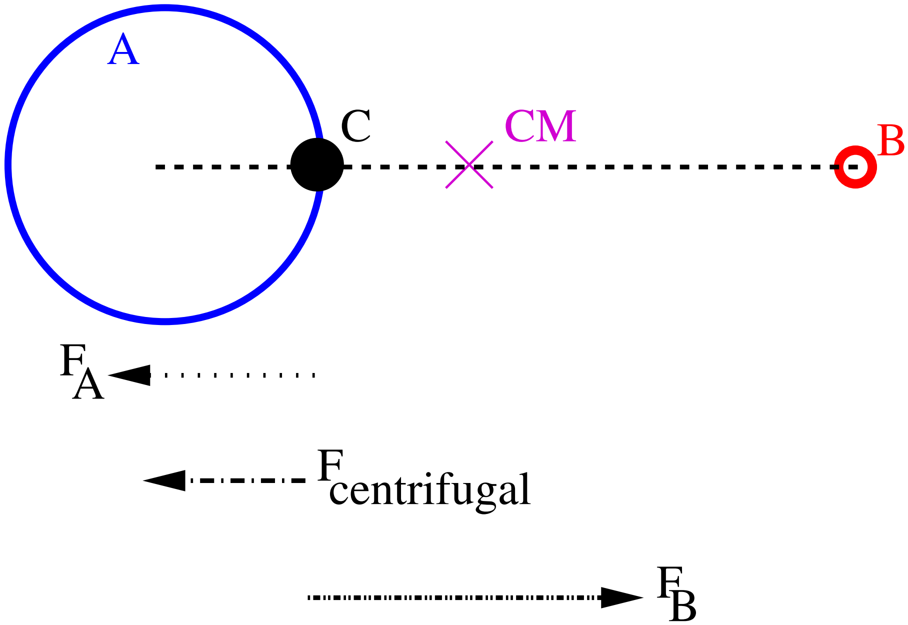


Diagram showing balance of forces in this binary system. F_A refers to the gravitational force felt at point C by star A, F_B the force felt at point C by star B, and $F_{\text{centrifugal}}$ the centrifugal force acting at point C due to its rotation.

- *Easy Way*: Both stars have mass 1 $M_\odot$. Therefore they will rotate about the center of the vector connecting the two stars' centers. If the separation is $2 \times 7 \times 10^{10} = 1.4 \times 10^{11}$ cm, then point C in the above diagram will lie on the center of mass, hence will undergo no rotation about the center of mass (no centrifugal force), and the gravitational force from A and B will cancel. Thus, $R = 1.4 \times 10^{11}$ cm.
- *Hard Way*: Assume a separation between the two stars of a distance R. The radius of star A is R_A. The center of mass is located at a distance $R/2$ between the two stars. Force balance gives the rotational frequency ω (balancing centrifugal acceleration with gravitational force).

$$\frac{1}{2}M\omega^2 R = \frac{GM^2}{R^2}$$

$$\omega^2 = \frac{2GM}{R^3}$$

In a frame moving with point C about the center of mass (CM), the balance of forces implies:

$$\frac{GM}{R_A^2} + \frac{2GM}{R^3}(R/2 - R_A) = \frac{GM}{(R - R_A)^2}$$

By inspection, one can show that if $R = 2R_A$, then $R - R_A = R_A$ and the above set of equations are satisfied. Thus again $R = 2R_A = 1.4 \times 10^{11}$ cm.

The Milky Way

1. Assume a galaxy is a disk. A *flat* rotation curve $V = R\Omega =$ constant implies that:
   (a) All the mass is concentrated near the center.
   (b) $M(R) \propto R^2$.
   (c) $M(R) \propto R$.
   (d) $M(R) \propto R^3$.

2. One bit radial and transverse velocity of objects near the sun (at distances much smaller than the distance of the sun to the galactic center).
   (a) is proportional to the distance to the object.
   (b) is proportional to the distance^2 to the object.
   (c) Is only a function of galactic longitude.
   (d) Always decreases as one moves away from the sun.

3. Which of the following was not a point of contention in the Curtis-Shapley debate about the distance to Andromeda?
   (a) Shapley had thought he had measured the proper motion of stars in M31.
   (b) The detection of Cepheids (stars with the same period-luminosity relation as Cepheids) implied that M31 was distant.
   (c) Novae were believed to be detected in M31 – if M31 was as distant as Curtis believed, then these would have to be brighter than novae in our galaxy.
   (d) Given the expected velocities, as seen from radial motion, then if M31 was close (distance to globular clusters) then one should see proper motions.

4. The extinction towards the galactic center is 28 magnitudes in the visual part of the spectrum. Assuming that the absorbing material is distributed uniformly between the sun and the galactic center (distance of 8 kpc), the extinction per parsec (in magnitudes)
   (a) 7×10^{-3} magnitudes pc^{-1}.
   (b) 3.5×10^{-3} magnitudes pc^{-1}.
   (c) 3.5 magnitudes pc^{-1}.
   (d) 7 magnitudes pc^{-1}.

5. The Oort limit, using the dispersion relation of stellar velocities in the solar neighborhood:
   (a) refers to the density of gas in the galactic disk from 21 cm hydrogen emission line spectroscopy.
   (b) Relates the total density of material in the disk from the vertical velocity dispersion of stars.
   (c) Relates the density of stars to the vertical velocity dispersion of stars.
   (d) Relates the density of all mater to the total velocity dispersion of stars.

Tidal Disruption of Stars in Galactic Center. Assume that the mass of the black hole in the galactic center is 10^7 M$_\odot$. Given that a star like the sun has a radius of 7×10^{10} cm, what is the minimum distance from the galactic black hole at which a sun-type star will be disrupted?
The tidal acceleration $a = gL/R$, where L is the size of the object. Take $L = 7 \times 10^{10}$ cm. Define the distance to the black hole to be D. Here we balance the tidal acceleration with the acceleration felt at the star's surface, where $M_\star$ is the mass of the star and $M_{\rm bh}$ is the mass of the black hole.

$$a = \frac{GM}{R^2}$$
$$\frac{GM_{\rm bh}}{D^3} R = \frac{GM_\star}{R^2}$$
$$D = R\left(M_{\rm bh}/M_\star\right)^{1/3} = 1.5 \times 10^{13}\ \text{cm}$$

Which is quite a small distance, on the order of 5×10^{-7} pc.

Interstellar Extinction (problem 22.3 in C+O). The globular cluster IAU C0923-545 has an integrated apparent visual magnitude of V = +13.0 and an integrated absolute visual magnitude of $M_V = -4.15$. It is located 9.0 kpc from Earth and is 11.9 kpc from the galactic centerm just 0.5 kpc south of the galactic midplane.

1. Estimate the amount of interstellar extinction between Earth and IAU C0923-545.
   The object is located 9000 pc away. The expected visual magnitude then becomes:

   $$V_E = M_V + 5\log_{10}(900) = 10.62$$

   The visual magnitude $V = 13.0$. Therefore the extinction $A = V - V_E = 2.38$.

2. What is the amount of interstellar extinction per kiloparsec?
   The visual extinction per kpc is $2.38/9\ \text{kpc}^{-1} = 0.264\ \text{mag kpc}^{-1}$.

Other Galaxies

1. Spiral arms are believed to be:
   (a) Winding of gas, dust, and stars due to the differential rotation of spiral arms.
   (b) Gravitational instabilities that result in a density wave which sweeps out through the galactic plane at constant angular speed.
   (c) From from the gravitational interaction between stars and gas alone within the galactic disk.
   (d) The accretion of gas onto galaxies in dense galactic clusters.
2. The fact that spiral arms are very thin in the disk implies that:
   (a) The stars undergo small velocity random motions about their much faster orbital motion.
   (b) They are very young and they have not had a chance to equilibrate.
   (c) External pressure balance keeps the stars, gas, and dust localized about the disk plane.
   (d) They spin so fast that any bulge flattens out into a disk.
3. The mass of galaxy clusters, from observations of their velocity dispersions and the use of the virial theorem, imply that:
   (a) The mass is dominated by the galaxies.
   (b) The mass is dominated by the intracluster gas.
   (c) The mass is dominated by dark matter.
   (d) New physics must be used to describe gravitational interactions over these long length scales.
4. The relaxation towards thermodynamic equilibrium of galaxies and globular clusters is due to:
   (a) Friction of stars with the hot gas and dust.
   (b) Gravitational interaction and scattering between stars and clouds.
   (c) Energy loss from photons.
   (d) Cooling of the gas through thermal emission.
5. Stars that deviate slightly from their equilibrium circular orbits in a spiral disk undergo:
   (a) Simple harmonic motion about their equilibrium position (like a pendulum or spring).
   (b) Linear (constant velocity) motion.
   (c) Random, stochastic motion due to frequent interactions with nearby stars and dust.
   (d) Epicyclic motion about their equilibrium positions.