Hints For Problem Set #6

1. (Shu 7.1): This is a nonrelativistic degenerate gas. The electron pressure $P_e = n_e p_x^2/m_e$. Use the uncertainty principle $p_x \sim h/\Delta x = h n_e^{1/3}$, where Δx is the average spacing between electrons, $v = (\Delta x)^3 = 1/n$ is the volume occupied per electron. Use these results to get the dimensionally correct expression for the pressure.

   For the second part, note that the mass density $\rho = A m_p n_+$, where n_+ is the ion number density. This is a neutral medium therefore $Z n_+ = n_e$ (the total positive charge density must equal the negative charge density). Express n_e in terms of ρ and m_p and substitute into the expression for electron pressure.

   For the third part, simply substitute in the thermal pressure into the above expression (technically, if the temperature is below the "Fermi temperature" T_f, the thermal pressure is given by another relation altogether). Compare the two pressures given a mass density $\rho = 10^6$ gm cm^{-3}.

   Finally, assume the ions have pressure given by degeneracy pressure. Go through the arguments, letting $P_+ = n_+ p_+^2/(A m_p)$ to calculate the ion degeneracy pressure as a function of ion number density n_+. Calculate the ratio of ion thermal pressure to ion degeneracy pressure:

$$\frac{P_+^{\text{thermal}}}{P_+^{\text{degeneracy}}} = \frac{n_+ kT}{0.0485 h^2 n_+^{5/3}/(A m_p)}$$

   To do this, it is useful to calculate the ion number density n_+ given that $A = 4$ and $\rho = 10^6$ gm cm^{-3}. Think about why the ion degeneracy pressure would be so much smaller at similar number densities?

2. (Shu 7.2): Note that the degeneracy pressure of nonrelativistic particles (whether due to electrons or ions) follows the equation of state $P \propto n^{5/3}$. Therefore the central electron pressure is a function of the central electron number density n_{ec}, which itself is a function of the central mass density ρ_c (which is already given in terms of M and R). Relate the central pressure calculated from virial arguments, $P_c = 0.770 GM^2/R^4$, and relate that to the pressure due to the central mass density, and solve for R.

   For the second part, use the result you found and calculate out the radius of a white dwarf with $M = 0.5\, M_\odot$ and $M = M_\odot$ and compare to the size of the earth (which has radius $R = 6 \times 10^8$ cm).

3. (Shu 7.3): Now consider an ultrarelativistic gas. The pressure $P = n v_x p_x = n c p_x$. Determine that $P_e \propto n_e^{4/3}$. This is the pressure of an ultrarelativistic degenerate electron gas. Equate this to the expression for the nonrelativistic degeneracy pressure ($P_e = 0.0485 h^2 n_e^{5/3}/m_e$ to determine the critical electron density n_e at which the gas becomes relativistic.

   Use the above expression for n_e to determine $v_x \sim h n_e^{1/3}/m_e$, and verify that this velocity is relativistic (of order c).

   For the final part, use $Z/A = 0.5$ to solve for R given the mass $M = M_\odot$. Using this density, compare to the density calculated given n_e as shown above.

4. (Shu 7.4): Perform a simple substitution to get the electron pressure in the relativistic regime as a function of mass density. Now note that:

$$P_c = 0.123 \frac{hc}{m_p^{4/3}} \left(\frac{Z}{A}\right)^{4/3} \rho_c^{4/3} = 11.0 \frac{GM^2}{R^4}$$

And given ρ_c in terms of M and R, you can show that R cancels out and that the mass of the star becomes a constant. Solve for M, which is M_{Ch} (the Chandrasekhar mass), given $Z/A = 0.5$.

5. (Shu 7.6): To calculate the mass-radius relation for a neutron star, replace m_e with m_p (the neutrons are now the degenerate gases). Furthermore, the atomic units are individual "neutrons" so $A = 1$ and the charge is irrelevant in determining the number density of neutrons.

6. (Shu 7.7): The rotational energy of a spinning sphere $E = \frac{1}{2}I\omega^2 = \frac{1}{5}MR^2\omega^2$. Given its mass and radius, determine its rotational energy. Finally, note that the power radiated out is given by:

$$P = -\frac{d}{dt}\left(\frac{1}{5}MR^2\omega^2\right) = -\frac{2}{5}MR^2\omega\dot{\omega}$$

Note that for the constant spin-down rate:

$$\dot{\omega} = -\omega_0/T$$

Where ω_0 is the current angular velocity and $T = 2500$ yr. To use useful quantities, convert T into seconds. Calculate the power P and compare to the luminosity of the Crab nebula ($L = 3 \times 10^{38}$ erg s^{-1}).

For the second part of the problem, suppose a body kept its mass M, its angular momentum J, and its shape (say from sphere into a smaller sphere) as it collapsed from R_1 to radius $R_2 < R_1$. Use the result that $J_1 = I_1\omega_1 = I_2\omega_2$. To determine I_2/I_1 as a function of R_2/R_1, take the simple case of a homogenous sphere of mass M and radius R. Finally, calculate out the ratio of new and old rotational energies:

$$\frac{E_2}{E_1} = \frac{I_2\omega_2^2}{I_1\omega_1^2}$$

As some function of R_2 and R_1. Now note that $R_2/R_1 = 10^6/10^9 = 10^{-3}$ for the collapse of a white dwarf into a neutron star. To determine where the energy for speed-up comes from, calculate the relative binding energies of white dwarf and neutron star.