Hints For Problem Set #3

Here are some hints to problem set #3. I will outline how to solve the problems and give other useful tips not within the problem set.

1. For part (a), the center of mass velocity is the mass-weighted average velocity of the system of particles. For a system of N particles with given masses and velocities M_i and $\mathbf{v}_i$, where $i = 1, \ldots N$, the center of mass velocity is:

$$\mathbf{v}_{\text{CM}} = \frac{\sum_{i=1}^{N} M_i \mathbf{v}_i}{\sum_{i=1}^{N} M_i}$$

The center of mass frame is useful because the total momentum of the system in that frame is zero.

For part (b), solve the problem by noting that in the center of mass frame, the motions of both particles are circles about their center of mass position: Their velocities in the center of mass frame

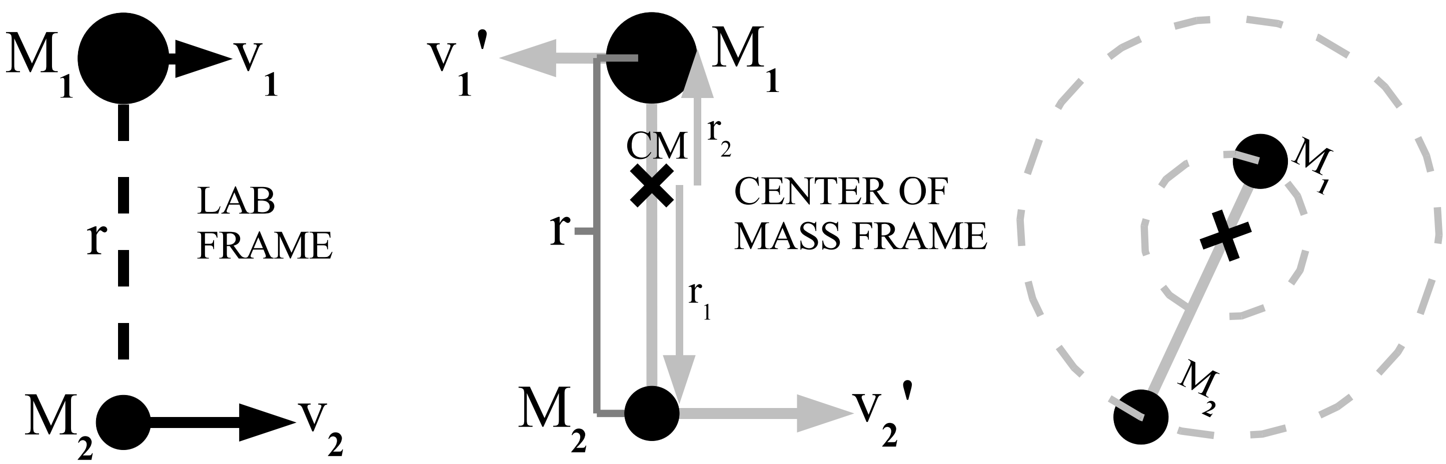

Figure 1: Velocities as seen in the lab frame and the center of mass frame, in which $\mathbf{v}_1' = \mathbf{v}_1 - \mathbf{v}_{\text{CM}}$ and $\mathbf{v}_2' = \mathbf{v}_2 - \mathbf{v}_{\text{CM}}$. Both masses undergo circular motion with radius r_1 and r_2 (for mass M_1 and M_2, respectively) about the center of mass.

are given by:

$$\mathbf{v}_1' = \mathbf{v}_1 - \mathbf{v}_{\text{CM}}$$
$$\mathbf{v}_2' = \mathbf{v}_2 - \mathbf{v}_{\text{CM}}$$

In the center of mass frame, each particle undergoes circular motion. Without loss of generality choose mass M_1. In some frame *rotating* with mass M_1, the gravitational force between them is balanced by the centrifugal force pushing mass M_1 out, therefore:

$$F_{\text{gravitational}} = F_{\text{centrifugal}}$$
$$\frac{GM_1M_2}{r^2} = \frac{M_1 {v_1'}^2}{r_1}$$

To solve this problem for r, determine what r_1 (separation of mass M_1 from the center of mass) and r_2 (separation of mass M_2 from the center of mass) are.

2. To solve the problem carefully, assume the carbon nucleus has infinite mass (in reality, the mass of the carbon atom is orders of magnitude larger than the electron mass). The angular momentum of this system is quantized in units of $\hbar$, Planck's constant divided by 2π:

$$\ell = m_e v r = n\hbar$$

Where n is some integer.

In a frame "moving" with the electron, the centrifugal force balances out the electrostatic attraction between the electron and the carbon nucleus, charge $Z = 6$:

$$\frac{m_e v^2}{r} = \frac{Ze^2}{r^2}$$

Use the quantization of angular momentum with the force balance equation to calculate v_n and r_n (electron velocities and radii for energy level n) – make sure to check that your answers have the proper dimension! The energy of the bound state E_n are given by:

$$E_n = \frac{1}{2} m_e v_n^2 - \frac{Ze^2}{r_n} = -\frac{Ze^2}{2r_n}$$

The bound state is $n = 1$. The energy needed to ionize a carbon atom with one electron is the energy required to unbound the electron in the ground state.

3. For part (a), the flux of radiation is the luminosity (power) over the surface area:

$$F = \frac{L}{4\pi r^2}$$

Recall that the energy flux is the flow rate of energy density, and has dimensions of energy density $\times$ the speed at which the energy is propagated. Knowing the speed at which electromagnetic energy propagates, and the flux, you can calculate the energy density u. Compare to the energy density due to blackbody radiation $u = aT^4$, using values for T for the cosmic microwave background and a as given in the book.

For part (b) note that the density of air $\rho = 10^{-3}$ gm cm^{-3} and that the angular frequency $\omega = 2\pi\nu$, with ν as a given. Compare to the energy density of the cosmic microwave background.

4. Here is a schematic diagram of the asteroid blasting off at some velocity $\mathbf{v}$ perpendicular to the plane of its initial orbit. I have drawn a before and after pictures here.

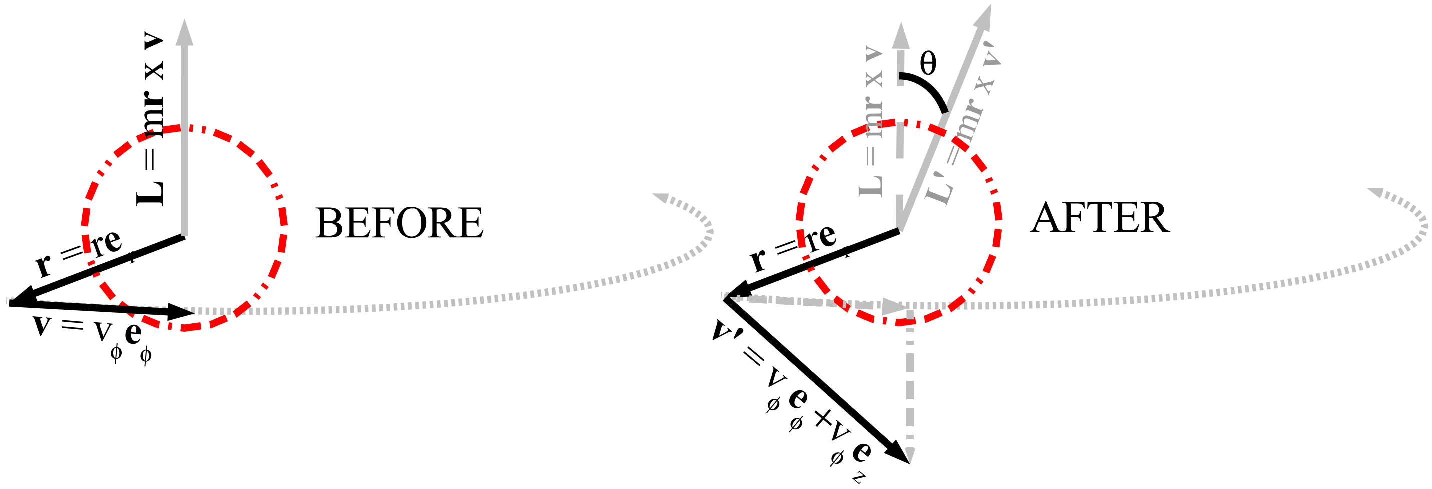

Figure 2: Plot of the orbital parameters $\mathbf{r} = r\mathbf{e}_r$, $\mathbf{v}$, and $\mathbf{L} = m\mathbf{r} \times \mathbf{v}$ before and after the orbital change. θ refers to the angle between the old $\mathbf{L}$ and newer $\mathbf{L}'$, and is the angle of inclination of the new orbit relative to the old orbit.

To determine whether the orbit is circular, elliptical, parabolic, or hyperbolic, note the following result:

$$\begin{aligned} E &> 0 && \text{hyperbolic} \\ E &= 0 && \text{parabolic} \\ E_{\text{min}} < E &< 0 && \text{elliptical} \\ E &= E_{\text{min}} && \text{circular} \end{aligned}$$

Here the E is calculated by keeping the angular momentum constant while changing the *total* (kinetic + potential) energy of the particle of mass m in its orbit. Note that for circular orbits of angular momentum magnitude $L = \sqrt{Gm^2Mr}$, the energy within the orbit is $E = E_{\text{min}} = -GMm/(2r) = -L^2/(2mr^2)$. Calculate out the total energy at one instant of its orbit:

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r}$$
$$E' = \frac{1}{2}mv'^2 - \frac{GMm}{r}$$

Before and after the orbital change. And calculate out the angular momentum before and after the orbit:

$$\mathbf{L} = m\begin{vmatrix} \mathbf{e}_r & \mathbf{e}_\phi & \mathbf{e}_z \\ r & 0 & 0 \\ 0 & \sqrt{\frac{GM}{r}} & 0 \end{vmatrix}$$
$$\mathbf{L}' = m\begin{vmatrix} \mathbf{e}_r & \mathbf{e}_\phi & \mathbf{e}_z \\ r & 0 & 0 \\ 0 & \sqrt{\frac{GM}{r}} & -\sqrt{\frac{GM}{r}} \end{vmatrix}$$

The inclination of the orbit is given by θ. Note the following result:

$$\mathbf{L} \cdot \mathbf{L}' = |\mathbf{L}'| \, |\mathbf{L}| \cos\theta$$

5. For part (a) assume the mass of the moon is zero (a good approximation, since $M_{\text{Moon}} \ll M_{\text{Earth}}$). Furthermore, the mass of the sun is much larger than that of the Earth, $M_{\text{Sun}} \gg M_{\text{Earth}}$. The angular frequency of the orbit is given by:

$$\omega = \sqrt{\frac{GM}{R^3}}$$

Where R is the radius of the orbit. Recall that $\omega = 2\pi/P$ where P is the orbital period.

For part (b) again assume the circular orbit but note that the energy of Earth in its orbit has changed. The velocity of Earth does not change before and after the sun's explosion.

$$E_{\text{old}} = \frac{1}{2} M_{\text{Earth}} v^2 - \frac{GM_{\text{Sun}} M_{\text{Earth}}}{R} = -\frac{GM_{\text{Sun}} M_{\text{Earth}}}{2R}$$

To be unbound, $E_{\text{new}} = 0$:

$$E_{\text{new}} = \frac{1}{2} M_{\text{Earth}} v^2 - \frac{GM'_{\text{Sun}} M_{\text{Earth}}}{R} = 0$$

And use these results to calculate the fraction of the sun's mass that needs to be lost for the earth to become unbound.