Practice Problems: ASTR 211 Final Exam

1. An object in the sky is located at celestial coordinates RA: $19^{h}\ 23^{m}\ 14^{s}$, Dec: -30^{ffi} 22’ 3”.

   (a) What is the latitude, north of which the object will not be on the horizon?
   Since the object is located at -30^{ffi} 22’ 3”, look for a northern latitude at which the object will not be visible below the horizon. At the equator, the maximum elevation is $90 - \left(30^{\text{ffi}}\,22'\,3''\right) = 59^{\text{ffi}}\,37'\,57''$. Therefore the maximum northern latitude, above which the star lies below the horizon, is at $59^{\text{ffi}}\,37'\,57''$. This is shown below:

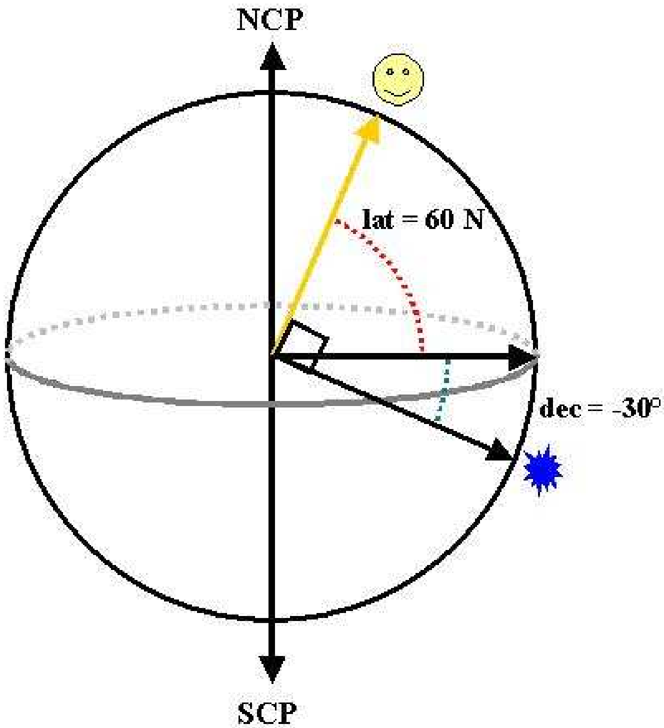


Figure 1: A stellar object at a declination of -30^{ffi} will be above the horizon except at a latitude of 30 N, where its maximum elevation will be the horizon.

   (b) At what times of the year will the object be visible in the night sky, where it can be observed? To answer this question, consider how the sun moves across the sky, and assume that the object is “visible” when the sun has a sidereal coordinate 6 hours ahead or behind this celestial object. Assume you are at zero degrees longitude.
   At the vernal equinox, March 21, the sun is located at RA: 0^{h} at solar noon. At the winter solistic, Dec. 20, the sun is at RA: 18^{h} at solar noon. Therefore, there remains approximately:

$$t = \frac{24 - 19^{h}\ 23^{m}\ 14^{s}}{24} \times 365.25 = 70.2 \text{ days}$$

   Before the vernal equinox. This is approximately at January 10. Therefore the object will be visible in the night sky from April 10 until October 10.

   (c) Assume you are in the Northern Hemisphere. On what date will the object have the highest elevation above the horizon at exactly local (solar) midnight? Assume you are at zero degrees longitude.
   At local solar midnight, the sun’s sidereal coordinate is at RA: $7^{h}\ 23^{m}\ 14^{s}$. Using the above result, or noting that this is exactly six months before when the sun is at RA: $19^{h}\ 23^{m}\ 14^{s}$, this date is June 10.

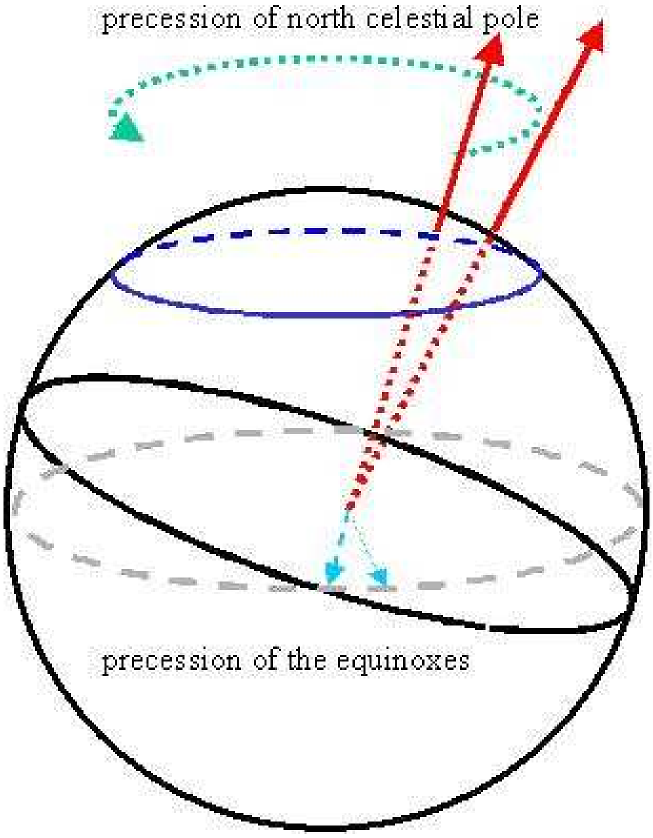


Figure 2: The precession of the celestial poles about the ecliptic. This also changes the location of the vernal equinox by approximately 54"/yr

2. The Zodiac consists of those constellations through which the sun passes through in its yearly motion across the sky. Equivalently, these are the constellations that lie on the ecliptic plane. The Zodiac constellations, and hence the signs that you see in horoscopes, were "finalized" 2500 years ago. Why is it that the Zodiac constellations no longer correspond to a specific Gregorian month (i.e., the sun is no longer in the constellation Leo in the month of August).
   The zodiac consists of constellations that lie on the ecliptic; therefore, as long as the earth's ecliptic plane does not change, the sun will continue to move across these constellations as seen from Earth. However, due to *precession* the locations of the equinoxes no longer correspond to their position 2500 years ago. In fact, the vernal equinox precesses by $360^{\text{ffi}}/23,500 = 0.92'$ per year.

3. The sun undergoes circular motion about the galactic center.

   (a) The period of the sun's orbit is 250 million years and the radius of the sun's orbit is 10 kiloparsecs. What is the sun's speed in its orbit around the galactic center?
   The speed of the sun's orbit:
   $$v_s = \frac{2\pi R}{T} = \frac{2\pi\,(10^4 \times 3.1 \times 10^{18}\ \text{cm})}{2.5 \times 10^8 \times 3.2 \times 10^7\ \text{s}} = 2.5 \times 10^7\ \text{cm s}^{-1}$$
   Or 250 km/s.

   (b) What is the mass interior to the sun's orbit?
   Assume a purely radial distribution of matter. Using Newton's laws of gravitation to determine

the gravitational force, where $G = 6.673 \times 10^{-8}$ in appropriate cgs units:

$$\frac{GM}{R^2} = \frac{v_s^2}{R}$$
$$M = v_s^2 R/G = 2.9 \times 10^{44}\ \text{gm} = 1.5 \times 10^{11}\ \text{M}_\odot$$

(c) Define a "disruptive" event (or collision) when a star comes within 100 AU of the sun. Given that the density of stars is 0.1 pc^{-3}, what is the probability of a disruptive event within one 250 million year period?

There are two questions to be asked. One, what is the volume swept out in this torus? If one

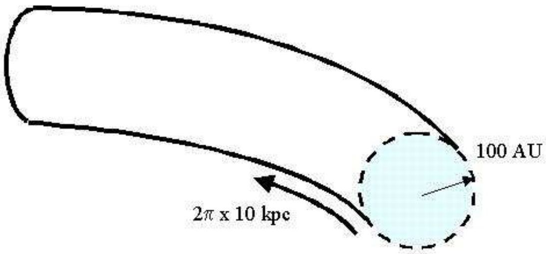


Figure 3: The volume filled out by the sun in its motion around the galaxy. The volume of the torus is $\pi R^2 d$, where R = 100 AU is the torus cross-section and $d = 2\pi \times 10$ kpc is the torus circumference.

or more stars lies within this torus, then the sun will be "disrupted" in its orbit. The volume of the torus swept out by the sun in one orbit is:

$$V = \pi R^2 d = \pi \left(100 \times 1.5 \times 10^{11}\ \text{cm}\right)^2 \times \left(10^4 \times 3.1 \times 10^{18}\ \text{cm}\right) = 2.2 \times 10^{53}\ \text{cm}^3 = 0.257\ \text{pc}^3$$

The density of stars in the solar neighborhood is $n = 0.1$ pc^{-3}. Therefore the probability of a collision:

$$p = nV = 2.57 \times 10^{-2}$$

4. An object from Earth is observed to have a synodic period of 2 years.

(a) What are the **two** possible orbital periods of the star?

If the planet is superior, then it moves with a slower speed. If it is inferior, the planet moves with faster angular speed. The earth has orbital frequency $\omega_\oplus = 2\pi/P_\oplus$. The planet has orbital frequency $\omega_P = 2\pi/P_P$. For either case, after a period of time T the relative angle (in radians) subtended by the planets:

$$\text{inferior:} \quad \theta = \omega_{\text{rel}} T = 2\pi T \left(\frac{1}{P_P} - \frac{1}{P_\oplus}\right)$$
$$\text{superior:} \quad \theta = \omega_{\text{rel}} T = 2\pi T \left(\frac{1}{P_\oplus} - \frac{1}{P_P}\right)$$

A synodic period is when the relative angle between the planets is 2π radians. Thus, setting $2\pi = \omega_{\text{rel}} P_{syn}$:

$$\text{inferior:} \quad \frac{1}{P_{\text{syn}}} = \frac{1}{P_P} - \frac{1}{P_\oplus}$$
$$\text{superior:} \quad \frac{1}{P_{\text{syn}}} = \frac{1}{P_\oplus} - \frac{1}{P_P}$$

The synodic period is 2 years. $P_\oplus = 1$ yr. Therefore we have the following periods:

$$P_{\text{inferior}} = \frac{2}{3} \text{ yr}$$
$$P_{\text{superior}} = 2 \text{ yr}$$

(b) Using Kepler's third law, what are the two possible orbital radii of these planets?
Kepler's third law states that $P^2 \propto a^3$, where P is the period and a is the semimajor axis. Assuming circular orbits, the radii of the inferior and superior planets:

$$R_{\text{inferior}} = (2/3)^{2/3} \text{ AU} = 0.76 \text{ AU}$$
$$R_{\text{superior}} = 2^{2/3} \text{ AU} = 1.59 \text{ AU}$$

5. What are the dimensions of the following physical constants? You may express your answer in terms of fundamentals ($M^\alpha L^\beta T^\gamma$, where M is mass, L is length, and T is time) or in terms of cgs units ($\text{gm}^\alpha \text{ cm}^\beta \text{ s}^\gamma$).

(a) h (Planck's constant):The units of Planck's constant are energy-time. Energy has units of M L^2 T^{-2}. Therefore $h \equiv ML^2T^{-1}$. In cgs units, gm cm^{-2} s^{-1}.

(b) G (Gravitational constant): Recall that GM^2/L has units of energy. Therefore:

$$GM^2L^{-1} \equiv ML^2T^{-2}$$
$$G \equiv M^{-1}L^3T^{-2}$$

Or in cgs units, gm^{-1} cm^3 s^{-2}.

(c) c (speed of light):The speed of light is a velocity, therefore has units of $c \equiv LT^{-1}$ or in cgs units, cm s^{-1}.

6. A sun-grazing comet has a period of 10^6 years. The point of closest approach of the comet is 800,000 km (5.3×10^{-4} AU).

(a) What is the semimajor axis of the comet, using Kepler's third law?
Kepler's third law, $P^2 \propto a^3$, implies that the semimajor axis, $a = (10^6)^{2/3} = 10^4$ AU.

(b) What is the maximal distance of the comet from the sun, in AU?
The semimajor axis $2a = r_a + r_p$, where r_a is the aphelion (farthest) distance and r_p is the perihilion (closest) distance. Using the above results, given r_p:

$$r_a = 2 \times 10^4 \text{ AU} - 5.3 \times 10^{-4} \text{ AU} \approx 2 \times 10^4 \text{ AU}$$

(c) Using conservation of total (kinetic + potential) energy and conservation of angular momentum, what is the ratio of kinetic energy at perihilion (closest approach) to aphelion (furthest approach)?
First, calculate the angular momentum. Aphelion and perihilion correspond to maximal radial distances, so at these points the radial velocity is zero. The velocity is purely tangential. The

angular momentum $\ell = mv_t r$, where v_t is the tangential velocity. Conservation of angular momentum tells us, since the speed at aphelion and perihilion is the tangential velocity:

$$mv_a r_a = mv_p r_p$$
$$v_a r_a = v_p r_p$$

The ratio of the kinetic energies are:

$$\rho = \frac{\frac{1}{2}mv_p^2}{\frac{1}{2}mv_a^2} = \frac{v_p^2}{v_a^2} = \frac{r_a^2}{r_p^2} = \left(\frac{2 \times 10^4}{5.3 \times 10^{-4}}\right)^2 = 1.42 \times 10^{15}.$$

7. A certain binary star system has a period of 50 years. The binary system has parallax of 0.01" and semimajor axis of 0.3".

   (a) What is the semimajor axis, in AU, of this binary system?
   One can explicitly calculate out the distance to the star in AU, and from that calculate the separation. However, one can implicitly use the definition of the parsec. An object 1 AU in size, located a distance 1 pc away, will subtend 1". If it is located 10 pc away, the object will subtend 0.1". From this alone, $a = 0.3/0.01 = 30$ AU.

   (b) What is the total mass, in $M_\odot$, of the two objects?
   The complete form of Kepler's third law goes as $P^2 \propto a^3/M_{\text{tot}}$. Given this proportionality relation (note: it is a good idea to look over proportionality relations), we can construct the following equality:

$$\left(\frac{P}{1 \text{ yr}}\right)^2 = \left(\frac{a}{1 \text{ AU}}\right)^3 \left(\frac{M_{\text{tot}}}{M_\odot}\right)^{-1}$$

   Substituting the period in years and the semimajor axis in AU:

$$\frac{M_{\text{tot}}}{M_\odot} = \frac{30^3}{50^2} = 10.8$$
$$M_{\text{tot}} = 10.8\, M_\odot$$

   (c) Companion A has semimajor axis of 0.05" as seen from Earth while companion B has semimajor axis of 0.25". What are their masses, in $M_\odot$?
   Using the lever principle, the masses of the companions:

$$M_A = \frac{0.25}{0.3} M_{\text{tot}} = 9\, M_\odot$$
$$M_B = \frac{0.05}{0.3} M_{\text{tot}} = 1.8\, M_\odot$$

8. A telescope has diameter D, and it collects light of wavelength λ. Derive, using the uncertainty principle, the angular resolution of this telescope.
   A quantum of electromagnetic radiation, a photon, obeys the uncertainty principle. This photon

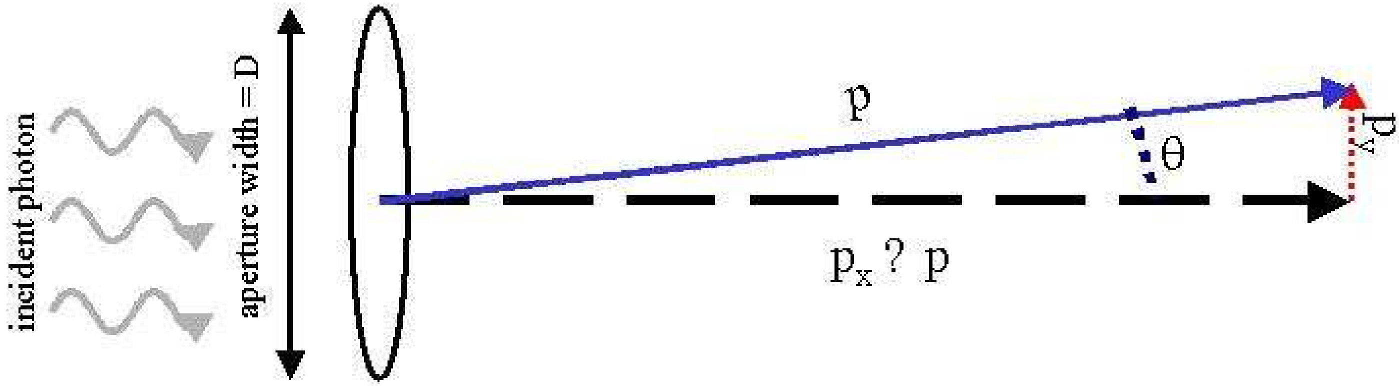


Figure 4: Uncertainty in the position of a given photon results in the uncertainty in the photon momentum, whose main contribution to the angle observed is along p_y. Here $\theta \approx p_y/p$.

obeys the uncertainty principle.

$$\Delta p \Delta x = h$$
$$\Delta p = \frac{h}{D}$$
$$\theta = \frac{\Delta p}{p} = \frac{h}{Dp}$$

The momentum of a photon $p = E/c = h/\lambda$, where λ is the wavelength. Therefore the angular resolution of a telescope with aperture D:

$$\theta = \frac{h\lambda}{Dh} = \frac{\lambda}{D}$$

9. Consider the light from two binary stars. One star has absolute magnitude $M_A = -2.5$, and the other star has absolute magnitude $M_B = 0$.

   (a) What is the absolute visual magnitude of the binary system?
   The magnitude of a star can be represented by the following:

   $$M_\star = -2.5 \log_{10}\left(\frac{F_\star}{F_0}\right)$$

   Where F_0 is the reference radiant flux, giving 0 magnitude. This implies that the magnitude of the two stars is given by:

   $$M_{A+B} = -2.5 \log_{10}\left(\frac{F_A + F_B}{F_0}\right) = -2.5 \log_{10}\left(10^{-0.4M_A} + 10^{-0.4M_B}\right)$$
   $$M_{A+B} = -2.5 \log_{10}(10 + 1) = -2.5 \log_{10} 11 = -2.603$$

(b) The stars are located 100 pc away. What is the apparent magnitude of each star and the binary as a whole?

The stars are 10 times further away, hence 100 times dimmer (100 times smaller radiant flux), hence 5 magnitudes larger each.

$$m_A = M_A + 5\log_{10}\left(\frac{d}{10\text{ pc}}\right) = 2.5$$
$$m_B = M_B + 5\log_{10}\left(\frac{d}{10\text{ pc}}\right) = 5$$
$$m_{A+B} = M_{A+B} + 5\log_{10}\left(\frac{d}{10\text{ pc}}\right) = 2.397$$

10. A star is observed to have a parallax of 0.2", a proper motion of 1.3"/year, and a Doppler blueshift of 1.1 Åfor a 5500 Å absorption line.

(a) What is the radial velocity of this star? Is it moving towards or away from us?

For small velocities, the radial velocity is given by:

$$\frac{v_r}{c} \approx \frac{\Delta\lambda}{\lambda} = -\frac{1.1}{5500}$$
$$v_r = -6 \times 10^6 \text{ cm/s}$$

The star has radial velocity of 6×10^6 cm/s towards us.

(b) What is the transverse velocity of this star?

Again, implicitly using the definition of a parsec, the star has transverse velocity:

$$v_T = \frac{1.3}{0.2} = 6.5 \text{ AU/yr} = \frac{6.5 \times 1.5 \times 10^{13}}{86400 \times 365.25} = 3.1 \times 10^6 \text{ cm/s}$$

(c) What is the star's total speed?

The transverse and radial velocities are perpendicular to each other. Therefore the total speed

$$v = \sqrt{v_r^2 + v_T^2} = 6.75 \times 10^6 \text{ cm/s}$$

(d) **bonus** Assuming the star's relative velocity to the sun does not change, what is the closest approach of the star to the sun? How many years will it take for the star to approach the sun?

To answer this question, first determine what angle the velocity vector makes with our line of sight. If one knows the law of cosines, it turns out that:

$$\cos\theta = \frac{v_T}{\sqrt{v_T^2 + v_R^2}}$$
$$\sin\theta = \frac{|v_R|}{\sqrt{v_T^2 + v_R^2}}$$

The closest approach distance:

$$y = d_0 \sin\theta = \frac{|v_R|\, d_0}{\sqrt{v_R^2 + v_T^2}}$$

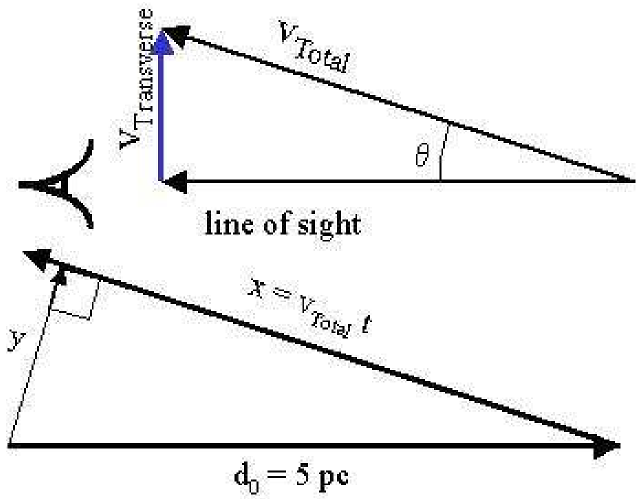


Figure 5: Diagram showing how to calculate the angle θ in determining the closest approach of the star. The geometry of closest approach is such that the hypotenuse has length d_0, the current distance to the star. $\sin\theta = y/d_0$.

And the time taken for closest approach:

$$\sqrt{v_R^2 + v_T^2}\,T = \frac{v_T d_0}{\sqrt{v_R^2 + v_T^2}}$$

$$T = \frac{v_T d_0}{v_R^2 + v_T^2}$$

Substituting the above values for the transverse velocity, the radial velocity, and the current separation of $d_0 = 5$ pc:

$$y = 4.45 \text{ pc}$$

$$T = 3.32 \times 10^4 \text{ yr}$$

11. Here are some questions about photons:

(a) What is the momentum carried by an individual photon of frequency ν?
A photon has energy $E = h\nu$. The momentum $p = E/c = h\nu/c$.

(b) Given a radiation flux of 30 erg s^{-1} cm^{-2}, what is the pressure, in dyne cm^{-2}?
Pressure is another way of saying *momentum flux*. Given the radiation flux, and the fact that for a collection of photons, the momentum of a photon $p = E/c$, implies that the pressure $P = F/c$. The pressure here:

$$P = \frac{30}{3 \times 10^{10}} = 10^{-9} \text{ dyne cm}^{-2}$$

(c) What is the force acting on a spherical dust grain of radius 10^{-4} cm?
The problem here is not uniquely specified. If we assume that the dust grain is perfectly absorbing, then each photon will impart momentum E/c onto the grain. The cross-sectional area of the grain is πR^2, where R is the grain radius. The force acting on the grain is then:

$$F = \pi R^2 P = 3.14 \times 10^{-17} \text{ dyne}$$

If the grain is perfectly reflecting, then each photon will bounce off perfectly from the grain with the same energy but opposite momentum (due to the fact that the rest-energy and momentum of the grain is very much larger than that of an individual photon). Thus, a momentum of $2E/c$ will be imparted to the grain by a photon of energy E. The force acting on the grain:

$$F = 2\pi R^2 P = 6.28 \times 10^{-17} \text{ dyne}$$

12. A hydrogen like atom has nuclear charge Ze and nuclear mass Am_H. Using the quantization of angular momentum ($L = n\hbar$) and the balance of centrigugal forces with electrostatic forces:

    (a) Calculate the radii of the different quantum states. What is the radius r_0 of the ground state? You may use A and Z as undetermined constants.
    The balance of centrifugal acceleration with electrostatic attraction:

    $$\frac{Ze^2}{r_n^2} = \frac{m_e v_n^2}{r_n}$$
    $$m_e^2 v_n^2 r_n^2 = m_e Z e^2 r_n$$

    And conservation of angular momentum:

    $$m_e v_n r_n = \ell_n = n\hbar$$
    $$m_e^2 v_n^2 r_n^2 = m_e Z e^2 r_n = n^2 \hbar^2$$
    $$r_n = \frac{n^2\hbar^2}{m_e e^2 Z} = a_0 n^2$$
    $$a_0 = 5.29 \times 10^{-8} Z^{-1} \text{ cm}$$

    (b) Calculate the energy levels of the different quantum states. What is the energy E_0 of the ground state? You may use A and Z as undetermined constants.
    Using the virial theorem, or from an explicit calculation, one can show that the energy levels go as:

    $$E_n = -\frac{Ze^2}{2r_n} = -\frac{Z^2 m_e e^4}{2\hbar^2 n^2} = E_0 n^{-2}$$
    $$E_0 = -13.6 Z^2 \text{ eV}$$

    You may assume the electron mass is much smaller than the nuclear mass.

13. Some interstellar dust has a visual wavelength opacity of 1 cm^2 gm^{-1} and a density of 10^{-18} gm cm^{-3}.

    (a) What is the optical depth assuming a thickness of 1 light year? What is the radiation flux after 1 light year assuming an incident intensity of 10^{-12} erg cm^{-2} s^{-1}?
    The mean free path $\ell = 1/(\kappa\rho)$. Therefore the optical depth:

    $$\tau = s/\ell = \kappa\rho s = \left(86400 \times 365.25 \times 3 \times 10^{10}\right)\left(10^{-18}\right) = 0.946728$$

And the outgoing intensity, given the incident intensity:

$$I_{\text{left}} = I_{\text{in}} e^{-\tau} = 3.88 \times 10^{-13} \text{ erg cm}^{-2} \text{ s}^{-1}$$

(b) What is the visual extinction, in magnitudes per light year, of this dusty cloud? If a star suffers visual extinction $A_V = 2$, what is the physical depth of this interstellar cloud?

The visual extinction, in terms of the optical depth, can be written as the following:

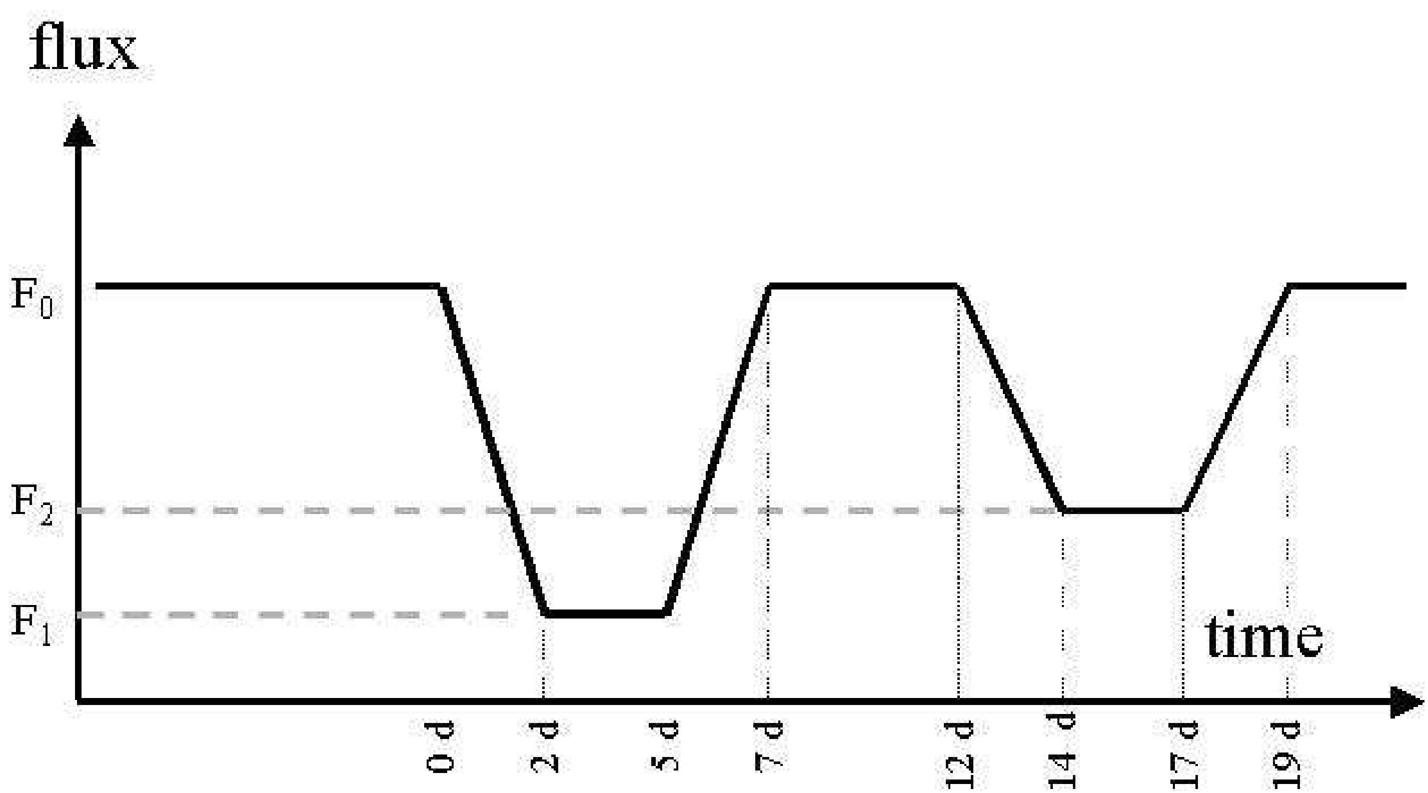
$$A = -2.5 \log_{10} \left(F_{\text{left}} / F_{\text{in}} \right) = -2.5 \log_{10} \left(e^{-\tau} \right) = -\frac{2.5 \ln \left(e^{-\tau} \right)}{\ln 10}$$

$$A = \frac{2.5}{\ln 10} \tau = \frac{2.5}{\ln 10} \times \kappa \rho s$$

$$A/s = \frac{2.5}{\ln 10} \kappa \rho = 1.09 \times 10^{-18} \text{ mag/cm} = 1.00 \text{ mag/ly}$$

Therefore, the dusty cloud needs to be 2 ly deep in order to cause a visual extinction $A_V = 2$.

14. The following edge-on binary has the following light curve, in visual magnitude. Assume the two stars have identical masses and move in circular orbits.



Let star A has smaller radius than star B. The stars move with velocity 10^7 cm s^{-1} about their center of mass at an inclination of 90ffi.

(a) What are the radii of star A and star B given the light curves shown above?

From the light curve going from maximum to first minimum, implies that we are seeing an eclipse of star A by star B. In the second, we are seeing a (partial) eclipse of star B by star

A. When the eclipse occurs, the stars are moving parallel to each other with a relative speed of 2×10^7 cm/s.

$$r_A = 2 \times 10^7 \times (2 \times 86400)/2 = 1.728 \times 10^{12} \text{ cm}$$
$$r_B = 2 \times 10^7 \times (5 \times 86400)/2 = 4.32 \times 10^{12} \text{ cm}$$

From which we have that $R_A = 0.4 R_B$.

(b) Assuming star A has twice the effective temperature as star B, what is F_1/F_0 and F_2/F_0 for the above stellar system?

When neither star is uneclipsed, the flux is given by:

$$F_0 = F_\star \left(R_A^2 T_A^4 + R_B^2 T_B^4\right)$$

At the first minimum, star A is eclipsed. The flux:

$$F_1 = F_\star \left(R_B^2 T_B^4\right)$$

And at the second minimum, star B is partially eclipsed by star A. The flux in this case:

$$F_2 = F_\star \left(R_A^2 T_A^4 + \left(R_B^2 - R_A^2\right) T_B^4\right)$$

Here $F_\star$ is a dimensional unit that gives the proper flux as seen from Earth. Thus:

$$\frac{F_1}{F_0} = \frac{R_B^2 T_B^4}{0.4^2 \times 16 T_B^4 R_B^2 + T_B^4 R_B^2} = \frac{1}{2.56 + 1} = 0.281$$
$$\frac{F_2}{F_0} = \frac{0.4^2 \times 2^4 R_B^2 T_B^4 + (1 - 0.4^2) R_B^2 T_B^4}{0.4^2 \times 2^4 R_B^2 T_B^4 + R_B^2 T_B^4} = \frac{2.56 + 1 - 0.16}{2.56 + 1} = 0.955$$

15. A star of mass $M_\star$ has density profile given by $\rho = \rho_0 (1 - r/R_\star)$, where $R_\star$ is the stellar radius.

(a) What is ρ_0 in terms of $M_\star$ and $R_\star$? Use the mass continuity equation:

$$\frac{dM}{dr} = 4\pi r^2 \rho(r)$$

From the continuity equation, we have that:

$$M_\star = \int_0^{R_\star} 4\pi\rho_0 (1 - r/R_\star) r^2 \, dr = 4\pi\rho_0 R_\star^3 \int_0^1 (1 - x) x^2 \, dx$$
$$M_\star = 4\pi\rho_0 R_\star^3 \left[\frac{x^3}{3} - \frac{1}{4}x^4\right]_0^1 = \frac{1}{3}\pi\rho_0 R_\star^3$$
$$\rho_0 = \frac{3M_\star}{\pi R_\star^3}$$

Where we have made the change of variable $x = r/R_\star$, so that $dr = R_\star \, dx$.

(b) What is the pressure $P(r)$ as a function of radius, setting $P(r = R_\star) = 0$? Write your answer in terms of $M_\star$ and $R_\star$. Recall the equation for hydrostatic equilibrium:

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

First, from the above equation we have that the mass of this star as a function of normalized radius $x = r/R_\star$:

$$M(x) = 4\pi\rho_0 R_\star^3 \int_0^x y^2 (1-y)\, dy = \frac{1}{3}\pi\rho_0 R_\star^3 \left(4\,[r/R_\star]^3 - 3\,[r/R_\star]^4\right)$$
$$M(x) = M_\star \left(4x^3 - 3x^4\right)$$

The pressure equation in terms of this normalized radius:

$$\frac{dP}{dx} = -\frac{3GM_\star^2}{\pi R_\star^4}\left(4x^3 - 3x^4\right)(1-x)\,x^{-2} = -\frac{3GM_\star^2}{\pi R_\star^4}\left(4x - 3x^2\right)(1-x)$$
$$P(x) = P_0 - \frac{3GM_\star^2}{\pi R_\star^4}\int_0^x \left(4y - 3y^2\right)(1-y)\, dy = P_0 - \frac{3GM_\star^2}{\pi R_\star^4}\int_0^x \left(4y - 7y^2 + 3y^3\right) dy$$
$$P(x) = P_0 - \frac{3GM_\star^2}{\pi R_\star^4}\left(2x^2 - \frac{7}{3}x^3 + \frac{3}{4}x^4\right)$$

Now at the surface, $x = 1$, $P(x) = 0$. Therefore the pressure is given by:

$$0 = P_0 - \frac{5GM_\star^2}{4\pi R_\star^4}$$
$$P_0 = \frac{5GM_\star^2}{4\pi R_\star^4}$$

And the pressure as a function of radius:

$$P(r) = \frac{5GM_\star^2}{4\pi R_\star^4} - \frac{3GM_\star^2}{\pi R_\star^4}\left(\frac{r}{R_\star}\right)^2\left(2 - \frac{7}{3}\left[\frac{r}{R_\star}\right] + \frac{3}{4}\left[\frac{r}{R_\star}\right]^2\right)$$

(c) Now replace $R_\star$ with $R_\odot = 7 \times 10^{10}$ cm and $M_\star$ with $M_\odot = 2 \times 10^{33}$ gm. What is the central pressure?
Substituting in the values for the sun, we get a central pressure $P_0 = 4.42 \times 10^{15}$ dyne cm^{-2}.

16. Neutrinos have a cross section for capture and interaction of approximately 10^{-41} cm^2 per particle.

(a) Given that the mass of the sun is $M_\odot = 2 \times 10^{33}$ gm, the radius $R_\odot = 7 \times 10^{10}$ cm, what is the column mass density of particles in a column from the sun's surface to the sun's core?
The mass column density is the mass of the sun distributed over a spherical shell of radius $R_\star$. Therefore, the column mass density:

$$\sigma = \frac{M_\star}{4\pi R_\star^4} = \frac{2 \times 10^{33}}{4\pi \times (7 \times 10^{10})^2} = 3.248 \times 10^{10} \text{ gm cm}^{-2}$$

(b) The average atomic mass of material in the sun is 2.2×10^{-24} gm/particle. What is the corresponding number density of particles in a column from the sun's surface to the sun's core?
With the above mass density, the column number density is given by:

$$\sigma_n = \frac{\sigma}{m} = \frac{3.248 \times 10^{10}}{2.2 \times 10^{-24}} = 1.48 \times 10^{34} \text{ particles cm}^{-2}$$

(c) What is the optical depth of the sun to neutrinos?

The optical depth $\tau = \sigma_n \sigma_\nu = 10^{-41} \times 1.48 \times 10^{34} = 1.48 \times 10^{-7}$. The sun is transparent to neutrinos.

17. Why are the angular resolutions of single-dish telescopes, say with a diameter of 100 meters, observing 1 cm wavelength sources, so much worse than optical telescopes, with diameters of 1 m, observing 5000 Å wavelength sources?

The angular resolution of a telescope is $\theta = \lambda/D$. For a typical radio telescope, the best resolution that can be achieved with a single dish is $\theta_{\rm radio} = 1/10^4 = 10^{-4}$ rad $= 10''$. For a typical optical telescope, the best resolution $\theta_{\rm optical} = 5 \times 10^{-7}$ rad $= 5.2'' \times 10^{-2}$.

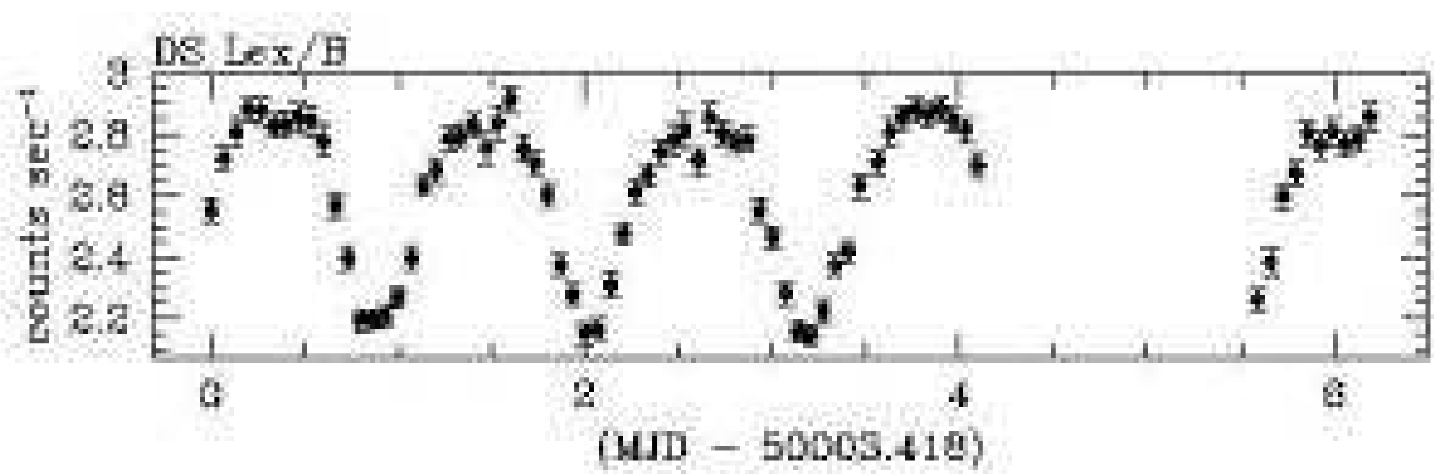
18. Describe the main mechanism (κ) that drives stellar pulsation. What is the ϵ mechanism, and why is it unimportant for stellar pulsation?

The κ mechanism results in *unstable* oscillations, due to the fact that when the He II partial ionization layers are compressed or heated adiabatically, their opacity *increases* (rather than decreases), resulting in a mild instability that can heat and push the outer layers of the star out. These partial ionization layers cool, and then drive the oscillation again.

Generally speaking, the variability is most pronounced when the He II partial ionization layer is located deeper in the stellar envelope rather than in the core. As a result more work can be done on the gas, resulting in greater luminosity and temperature variations.

The ϵ mechanism is where the compressions and expansions of the star result in variability in the energy generation rate, due explicitly to changes in density and pressure in a radiative nuclear-burning stellar core. This is not deemed important, because it is believed that stellar pulsation is dominated by pressure modes – as the frequency and wavenumber of these modes *increase*, they become more concentrated about the surface. In fact, the pressure modes that are characteristic of stellar pulsation have a node (zero displacement) at the core, implying that changes in core density and pressure are negligible – the ϵ mechanism is unimportant.

19. Shown below is the light curve of a typical white dwarf. This variability has period of 1.15 days.



The sound crossing time within a white dwarf is approximately 10 seconds. Is the pulsation due to pressure variations? Why or why not?

The fundamental pressure mode (breathing mode) has a period equal to the sound crossing time, here 10 seconds. All other pressure modes, those that possess nodes or that have angular profile, have higher frequencies and hence lower periods.

Gravity modes, in which the mode propagates in directions normal to the gravitational force (surface waves are a class of gravity modes), have smaller frequencies, hence longer periods, as their wavenumber increases. For gravity modes, the *shortest* period is the sound-crossing time. The period of oscillation in luminosity of this white dwarf is much longer than the sound-crossing time, implying strongly that this is not a pressure mode and, due to characteristic frequencies in the spectrum, is believed to be dominated by gravity modes.

20. A perfectly conducting interstellar cloud is threaded with a magnetic field.

    (a) How does the magnetic field and magnetic pressure scale with the size R of the cloud, assuming that magnetic flux is frozen?
    For a perfectly conducting fluid, the magnetic flux is fixed. Thus, we have that $BR^2 \equiv$ constant. From this we get that:

    $$B \propto R^{-2}$$
    $$p_B = \frac{B^2}{8\pi} \propto R^{-4}$$

    (b) If we assume adiabatic collapse of the cloud, such that $P \propto \rho^{5/3}$, where ρ is the cloud density, how does the pressure scale with the size R of the cloud?
    Assuming free-fall adiabatic collapse, the density $\rho \propto R^{-3}$. The pressure within the cloud goes as R^{-5}, assuming adiabatic free-fall collapse.

    (c) Based on what you have found, will magnetic fields eventually halt the free-fall collapse of this cloud? Why or why not?
    Gravity is attractive, but the gas pressure and magnetic pressure are both repulsive. However, if we assume that the gas remains adiabatic when collapse is finally halted, it is not possible for the magnetic pressure (in this instance) to halt the collapse – the magnetic pressure increases as the cloud size decreases, but the gas pressure increases faster.

21. From the equation of state of a relativistic degenerate Fermi gas, $P = K\rho^{4/3}$, show that these stars have a fixed mass and undefined radius – that is, show that the mass is given by a constant value independent of the radius of the star. Take K to be something dependent only on physical constants (Planck's constant, the speed of light, etc.) and independent of the star's properties.
    From the mass continuity equation, one gets that:

    $$\frac{dM}{dr} = 4\pi r^2 \rho$$

    From a dimensional argument, we can let:

    $$M \simeq R^3 \rho$$

    Now from the expression for hydrostatic equilibrium:

    $$\frac{dP}{dr} = -\frac{GM\rho}{r^2}$$

That an estimate for the pressure is given by the following:

$$P \simeq \frac{GM\rho}{R} = K\rho^{4/3}$$
$$\frac{GM}{RK} = \rho^{1/3} = \frac{M^{1/3}}{R}$$
$$M^{2/3} = K/G$$
$$M = (K/G)^{3/2}$$

In the above equation, the radius has "fallen out" – the star's structure is independent of the radius and has a fixed mass. A specific example is the Chandrasekhar limit for white dwarfs and neutron stars, and the Eddington radiative limit for very massive, luminous stars.

22. What gives rise to the Gamow peak in nuclear reactions? Why is quantum mechanical tunneling required to explain the reaction rate in stars? The height of the potential barrier is approximately 50 MeV, and the plasma temperature $T \sim 10^7$ K.
The Gamow peak arises from the quantum mechanical tunneling by particles, with much smaller kinetic energy than the barrier energy, across the barrier. A typical thermal energy of particles in the center of the sun is 1 keV $\ll$ 50 MeV. Therefore, without including the effects of quantum mechanics, there would be no nuclear reactions at the temperatures and densities at the sun's core.

The tunneling probability across the barrier goes as $p \propto \exp\left(-bE^{1/2}\right)$ (i.e., increases as the energy E increases), while the particle distribution goes as $E^{1/2}\exp\left(-E/k_B T\right)$ for all E. This Gamow peak appears at the tail of the particle distribution, where the tunneling probability is increasing, and manifests itself as a very small, yet the only source, of nuclear reactivity in the plasma at these low temperatures.

23. The solar wind has a density of 10 protons cm^{-3} and an average speed of 500 km s^{-1} at 1 AU.

   (a) What is the proton flux at 1 AU? Your answer should be in units of cm^{-2} s^{-1}.
   Proton flux $F_p = nv = 10 \times (5 \times 10^7) = 5 \times 10^8$ protons cm^{-2} s^{-1}.

   (b) What is the mass loss rate, in $M_\odot$ yr^{-1}, due to the solar wind?
   The number rate loss of protons due to the solar wind:

   $$\dot{M}_{SW} = 4\pi R^2 F_p m_H$$

   We take $R = 1$ AU $= 1.5 \times 10^{13}$ cm. The mass of a proton $m_H = 1.66 \times 10^{-24}$ gm. Therefore the mass loss rate:

   $$\dot{M}_{SW} = 2.35 \times 10^{12} \text{ gm s}^{-1} = 7.41 \times 10^{19} \text{ gm s}^{-1} = 3.7 \times 10^{-11} M_\odot \text{ yr}^{-1}$$

   (c) If we assume that the particle velocity does not change as we change distances from the sun, how does number density scale with d, the distance to the sun?
   No particles are being created or destroyed. Therefore $\dot{M}_{SW}$ is independent of radius. Furthermore, since $\dot{M}_{SW} = 4\pi R^2 n v m_H$, and if v is a constant, then we have that $n \propto R^{-2}$.

24. The sun has mass of $M_\odot = 2 \times 10^{33}$ gm and a radius of $R_\odot = 7 \times 10^{10}$ cm.

(a) What is the gravitational acceleration at the sun's surface?

$$g_\odot = \frac{GM_\odot}{R_\odot^2} = 2.73 \times 10^4 \text{ cm s}^{-2}$$

(b) What is the maximum angular frequency Ω at which the star will break up? That is, find the Ω such that the centrifugal acceleration balances out the gravitational acceleration.
Balance out the centrifugal acceleration with the gravitation acceleration:

$$\frac{GM_\star}{R_\star^2} = \Omega^2 R_\star$$
$$\Omega = \sqrt{\frac{GM_\star}{R_\star^3}} = 6.24 \times 10^{-4} \text{ s}^{-1}$$

(c) What period does this correspond to?

$$\Omega = \frac{2\pi}{P}$$
$$P = \frac{2\pi}{\Omega} = 10^4 \text{ s}$$

25. A proton, with rest mass of 980 MeV,is travelling at 0.99999 c, where c is the speed of light, relative to an observer on Earth.

(a) What is the total energy (kinetic + rest mass) of the photon as seen from Earth?
The total energy:

$$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}} \approx \frac{mc^2}{\sqrt{2\delta v/c}} = \frac{980 \text{ MeV}}{\sqrt{2 \times 10^{-6}}} = 693 \text{ GeV}$$

Where $\delta v = c - v$. Here we used the following Taylor expansion:

$$\left(1 - v^2/c^2\right)^{1/2} = \left(1 - (1 - \delta v/c)^2\right)^{1/2} = \left(2\delta v/c - \delta v^2/c^2\right)^{1/2} \approx \sqrt{2\delta v/c}$$

(b) What is the momentum of the photon as seen from Earth?

$$p = \frac{mv}{\sqrt{1 - v^2/c^2}} \approx \frac{mc}{\sqrt{2\delta v/c}} = \frac{980 \text{ GeV/c}}{\sqrt{2 \times 10^{-6}}} = 693 \text{ GeV/c}$$

Note that for photons of energy, say, 1 GeV, its momentum is 1 GeV/c.

(c) The sun emits radiation at a characteristic wavelength of 5000 Å. What is the wavelength and energy of this photon in the rest frame of the proton? Assume that the photon arrives head-on

to the photon.
In the frame of the proton, the radiation will be heavily blueshifted according to this formula:

$$\frac{\lambda}{\lambda_0} = \sqrt{\frac{1 - v/c}{1 + v/c}} \approx \sqrt{\frac{\delta v}{2c}}$$
$$\lambda = 3.89 \text{ Å}$$

26. The energy levels of the neutral hydrogen atom go as $E_n = -13.6 \text{ eV}/n^2$. The degeneracy of the energy levels goes as $2n^2$. At $T = 1000$ K:

    (a) What are the relative occupancies of the $n = 2$ and $n = 3$ levels relative to $n = 1$?
    Denote f_1, f_2, and f_3 as the occupancies of the $n = 1$, $n = 2$, and $n = 3$ states, respectively. Then:

    $$\frac{f_2}{f_1} = \frac{2 \times 2^2 \exp\left(\frac{E_0}{2^2 k_B T}\right)}{2 \times \exp\left(\frac{E_0}{k_B T}\right)} = 1.57 \times 10^{-51}$$

    For the $n = 3$ state:

    $$\frac{f_3}{f_1} = \frac{2 \times 3^2 \exp\left(\frac{E_0}{3^2 k_B T}\right)}{2 \times \exp\left(-\frac{E_0}{k_B T}\right)} = 1.07 \times 10^{-60}$$

    (b) Assuming the density of hydrogen is $\rho = 10^{-3}$ gm cm^{-3}. What are number densities of hydrogen in the neutral state, in the $n = 1$ state, and in the $n = 2$ state?
    In this approximation, we can take the number density of the $n = 1$ state as:

    $$n_1 = \frac{\rho}{m_H} = 6.022 \times 10^{20} \text{ cm}^{-3}$$

    The number densities of $n = 2$ and $n = 3$ neutral hydrogen atoms:

    $$n_2 \approx \frac{f_2}{f_1} n_1 = 9.45 \times 10^{-31} \text{ cm}^{-3}$$
    $$n_3 \approx \frac{f_3}{f_1} n_1 = 6.44 \times 10^{-40} \text{ cm}^{-3}$$

27. Consider the following about degenerate gases:

    (a) Using the uncertainty principle, show that the average energy of a particle in a nonrelativistic degenerate gas is given by the following: $E \sim \frac{\hbar^2 n^{2/3}}{2m}$, where n is the particle number density and m is the particle mass.
    The average spacing between particles of number density n is $\Delta x = n^{-1/3}$. Using the uncertainty principle:

    $$\Delta p n^{-1/3} = \hbar$$
    $$\Delta p = \hbar n^{1/3}$$

The representative kinetic energy of the nonrelativistic particles is $\epsilon \sim (\Delta p)^2 / (2m)$. Therefore the particle energy:

$$\epsilon \sim \frac{(\Delta p)^2}{2m} = \frac{\hbar^2 n^{2/3}}{2m}$$

(b) Show that the pressure in a nonrelativistic degenerate gas goes as $n^{5/3}$.
The pressure scales as the energy density. Then energy density goes as $E \sim \epsilon n$, so that $P \sim \epsilon n$.

$$E \sim \epsilon n \sim \frac{\hbar^2}{2m} n^{5/3}$$

(c) Using the above expression for the energy, find the critical number density n at which a degenerate electron gas becomes relativistic.
To find the number density at which a degenerate Fermi gas becomes relativistic, set the average kinetic energy to the rest energy of the particle:

$$\epsilon = mc^2 = \frac{\hbar^2 n^{2/3}}{2m}$$
$$n = 2^{3/2} (mc/\hbar)^3$$

The result $n = (mc/\hbar)^3$ is also acceptable.

28. Consider the opacity of two fully radiative stars. The opacity of star B is 10 times larger than the opacity of star A.

(a) If it takes a time τ for photons to radiate out of star A, how many times τ does it take for photons to radiate out of star B?
Radiative diffusion of photons is a “random walk” process. If everything else stays the same but the opacity increases by a factor of 10, then the mean free path decreases by a factor of 10. This requires $10^2 = 100$ more steps to reach the star’s surface. Therefore it takes 100τ for a photon to radiate out through the surface.

(b) Assuming both stars have the same internal energy (same radius, same mass, and same internal structure). How much larger is the luminosity of star A relative to star B? Use dimensional analysis to determine the answer.
Assuming the stars have the same internal energy. Assume radiation is the only way to transport energy in both stars. For the star with opacity 10 times higher, A given photon will leak out over 100 times the length of the lower-opacity star. Therefore, the luminosity of the opaque star will be 100 times lower.